Unit 4 Planning the Unit

In this unit, students study arithmetic and geometric sequences and implicit and explicit rules for defining them. Then they analyze exponential and logarithmic patterns and graphs as well as properties of logarithms. Finally, they solve exponential and logarithmic equations.

Vocabulary Development

The key terms for this unit can be found on the Unit Opener page. These terms are divided into Academic Vocabulary and Math Terms. Academic Vocabulary includes terms that have additional meaning outside of math. These terms are listed separately to help students transition from their current understanding of a term to its meaning as a mathematics term. To help students learn new vocabulary:

- Have students discuss meaning and use graphic organizers to record their understanding of new words.
- Remind students to place their graphic organizers in their math notebooks and revisit their notes as their understanding of vocabulary grows.
- As needed, pronounce new words and place pronunciation guides and definitions on the class Word Wall.

Embedded Assessments

Embedded assessments allow students to do the following:

- Demonstrate their understanding of new concepts.
- Integrate previous and new knowledge by solving real-world problems presented in new settings.

They also provide formative information to help you adjust instruction to meet your students' learning needs.

Prior to beginning instruction, have students unpack the first embedded assessment in the unit to identify the skills and knowledge necessary for successful completion of that assessment. Help students create a visual display of the unpacked assessment and post it in your class. As students learn new knowledge and skills, remind them that they will be expected to apply that knowledge to the assessment. After students complete each embedded assessment, turn to the next one in the unit and repeat the process of unpacking that assessment with students.

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CollegeBoard

AP / College Readiness

Unit 4 continues to develop students' understanding of functions and their inverses by:

- Graphing exponential and logarithmic functions.
- Applying properties of exponents to develop properties of logarithms.
- Solving exponential and logarithmic equations.

Unpacking the Embedded Assessments

The following are the key skills and knowledge students will need to know for each assessment.

Embedded Assessment 1

Sequences and Series, *The Chessboard Problem*

- Identifying terms in arithmetic and geometric sequences
- Identifying common differences and common ratios
- Writing implicit and explicit rules for arithmetic and geometric sequences

Embedded Assessment 2

Exponential Functions and Common Logarithms, *Whether or Not*

- Examining exponential patterns and functions
- Identifying and analyzing exponential graphs
- Transforming exponential functions
- Graphing and transforming natural base exponential functions
- Examining common logarithmic functions
- Understanding properties of logarithms

Embedded Assessment 3

Exponential and Logarithmic Equations, *Evaluating Your Interest*

- Solving exponential equations
- Solving logarithmic equations
- Solving real-world applications of exponential and logarithmic functions

Suggested Pacing

The following table provides suggestions for pacing using a 45-minute class period. Space is left for you to write your own pacing guidelines based on your experiences in using the materials.

	45-Minute Period	Your Comments on Pacing
Unit Overview/Getting Ready	1	
Activity 19	3	
Activity 20	3	
Embedded Assessment 1	1	
Activity 21	5	
Activity 22	4	
Embedded Assessment 2	1	
Activity 23	3	
Activity 24	4	
Embedded Assessment 3	1	
Total 45-Minute Periods	**26**	

Additional Resources

Additional resources that you may find helpful for your instruction include the following, which may be found in the Teacher Resources at SpringBoard Digital.

- Unit Practice (additional problems for each activity)
- Getting Ready Practice (additional lessons and practice problems for the prerequisite skills)
- Mini-Lessons (instructional support for concepts related to lesson content)

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Series, Exponential and Logarithmic Functions

4

Unit Overview
In this unit, you will study arithmetic and geometric sequences and series and their applications. You will also study exponential functions and investigate logarithmic functions and equations.

Key Terms
As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Math Terms
- sequence
- arithmetic sequence
- common difference
- recursive formula
- explicit formula
- series
- partial sum
- sigma notation
- geometric sequence
- common ratio
- geometric series
- finite series
- infinite series
- sum of the infinite geometric series
- exponential function
- exponential decay factor
- exponential growth factor
- asymptote
- logarithm
- common logarithm
- logarithmic function
- natural logarithm
- Change of Base Formula
- exponential equation
- compound interest
- logarithmic equation
- extraneous solution

ESSENTIAL QUESTIONS
- How are functions that grow at a constant rate distinguished from those that do not grow at a constant rate?
- How are logarithmic and exponential equations used to model real-world problems?

EMBEDDED ASSESSMENTS
This unit has three embedded assessments, following Activities 20, 22, and 24. By completing these embedded assessments, you will demonstrate your understanding of arithmetic and geometric sequences and series, as well as exponential and logarithmic functions and equations.

Unit Overview
Ask students to read the unit overview and mark the text to identify key phrases that indicate what they will learn in this unit.

Key Terms
As students encounter new terms in this unit, help them to choose an appropriate graphic organizer for their word study. As they complete a graphic organizer, have them place it in their math notebooks and revisit as needed as they gain additional knowledge about each word or concept.

Essential Questions
Read the essential questions with students and ask them to share possible answers. As students complete the unit, revisit the essential questions to help them adjust their initial answers as needed.

Unpacking Embedded Assessments
Prior to beginning the first activity in this unit, turn to Embedded Assessment 1 and have students unpack the assessment by identifying the skills and knowledge they will need to complete the assessment successfully. Guide students through a close reading of the assessment, and use a graphic organizer or other means to capture their identification of the skills and knowledge. Repeat the process for each Embedded Assessment in the unit.

Developing Math Language
As this unit progresses, help students make the transition from general words they may already know (the Academic Vocabulary) to the meanings of those words in mathematics. You may want students to work in pairs or small groups to facilitate discussion and to build confidence and fluency as they internalize new language. Ask students to discuss new academic and mathematics terms as they are introduced, identifying meaning as well as pronunciation and common usage. Remind students to use their math notebooks to record their understanding of new terms and concepts.

As needed, pronounce new terms clearly and monitor students' use of words in their discussions to ensure that they are using terms correctly. Encourage students to practice fluency with new words as they gain greater understanding of mathematical and other terms.

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UNIT 4
Getting Ready

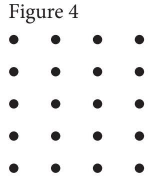
Use some or all of these exercises for formative evaluation of students' readiness for Unit 4 topics.

Prerequisite Skills

- Pattern recognition (Items 1, 2, 3) 7.NS.A.3
- Properties of exponents (Items 4, 5, 6) 8.EE.A.1
- Solving equations (Item 7) HSA-REI.B.3
- Writing and graphing functions (Item 8) HSA-IF.B.4, HSA-BF.A.1a

Answer Key

1. The numbers increase by consecutive increasing odd numbers.
2. 2, -4, 3
3. Figure 4



Sample explanation: Each figure has the same number of columns of dots as the figure number, and the number of rows of dots is always one more than the figure number.

4. a. $\frac{36x^4}{y^6}$
   b. $6a^2b^4$
   c. $2a^9b^8$
5. $3^4 = 81$
6. 8.7×10^5
7. $x = 2$
8. $C(t) = 2.5 + 0.5t$; slope $= 0.5$; y-intercept $= 2.5$

UNIT 4

Getting Ready

Write your answers on notebook paper. Show your work.

1. Describe the pattern displayed by 1, 2, 5, 10, 17, ...
2. Give the next three terms of the sequence 0, -2, 1, -3, ...
3. Draw Figure 4, using the pattern below. Then explain how you would create any figure in the pattern.

Figure 1 Figure 2 Figure 3

4. Simplify each expression.
   a. $\left(\frac{6x^2}{y^3}\right)^2$
   b. $(2a^2b)(3b^3)$
   c. $\frac{10a^{12}b^6}{5a^3b^{-2}}$
5. Evaluate the expression.
   $$\frac{3^{327}}{3^{323}}$$
6. Express the product in scientific notation.
   $$(2.9 \times 10^3)(3 \times 10^2)$$
7. Solve the equation for x.
   $$19 = -8x + 35$$
8. Write a function $C(t)$ to represent the cost of a taxicab ride, where the charge includes a fee of \$2.50 plus \$0.50 for each tenth of a mile t. Then give the slope and y-intercept of the graph of the function.

Getting Ready Practice

For students who may need additional instruction on one or more of the prerequisite skills for this unit, Getting Ready practice pages are available in the Teacher Resources at SpringBoard Digital. These practice pages include worked-out examples as well as multiple opportunities for students to apply concepts learned.

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Arithmetic Sequences and Series

ACTIVITY 19

Arithmetic Alkanes

Lesson 19-1 Arithmetic Sequences

My Notes

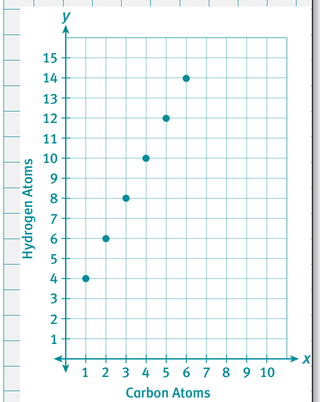
Learning Targets:

- Determine whether a given sequence is arithmetic.
- Find the common difference of an arithmetic sequence.
- Write an expression for an arithmetic sequence, and calculate the *n*th term.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Look for a Pattern, Summarizing, Paraphrasing, Vocabulary Organizer

Hydrocarbons are the simplest organic compounds, containing only carbon and hydrogen atoms. Hydrocarbons that contain only one pair of electrons between two atoms are called alkanes. Alkanes are valuable as clean fuels because they burn to form water and carbon dioxide. The number of carbon and hydrogen atoms in a molecule of the first six alkanes is shown in the table below.

Alkane	Carbon Atoms	Hydrogen Atoms
methane	1	4
ethane	2	6
propane	3	8
butane	4	10
pentane	5	12
hexane	6	14



1. **Model with mathematics.** Graph the data in the table. Write a function *f*, where $f(n)$ is the number of hydrogen atoms in an alkane with *n* carbon atoms. Describe the domain of the function.
   $f(n) = 2n + 2$; *n* is a positive integer.

Any function where the domain is a set of positive consecutive integers forms a ***sequence***. The values in the range of the function are the *terms* of the sequence. When naming a term in a sequence, subscripts are used rather than traditional function notation. For example, the first term in a sequence would be called a_1 rather than $f(1)$.

Consider the sequence {4, 6, 8, 10, 12, 14} formed by the number of hydrogen atoms in the first six alkanes.

2. What is a_1? What is a_3?
   $a_1 = 4$; $a_3 = 8$

3. Find the differences $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$, $a_5 - a_4$, and $a_6 - a_5$.
   Each difference is 2.

Sequences like the one above are called *arithmetic sequences.* An ***arithmetic sequence*** is a sequence in which the difference of consecutive terms is a constant. The constant difference is called the ***common difference*** and is usually represented by *d*.

MATH TERMS

A **sequence** is an ordered list of items.

WRITING MATH

If the fourth term in a sequence is 10, then $a_4 = 10$.

Sequences may have a finite or an infinite number of terms and are sometimes written in braces { }.

Common Core State Standards for Activity 19

HSA-SSE.B.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.★
HSF-BF.A.1	Write a function that describes a relationship between two quantities.★
HSF-BF.A.1a	Determine an explicit expression, a recursive process, or steps for calculation from a context.
HSF-BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

ACTIVITY 19

Guided

Activity Standards Focus

In Activity 19, students learn to identify arithmetic sequences and to determine the *n*th term of such sequences using recursive and explicit formulas. They also write formulas for the sum of the terms in arithmetic sequences, known as an arithmetic series, and calculate the *n*th partial sums of arithmetic series. Finally, they represent arithmetic series using sigma notation and determine the sums. There is a lot of notation in this activity, and students may get lost in the symbols. Encourage students to read carefully, and check often to be sure they can explain the meanings of the formulas and the variables within the formulas.

Lesson 19-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2–3 #4–5
Check Your Understanding
#9–10 #11–14 #15–16
Example A
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to describe the pattern and give the next 3 terms of the following sequences.

1. $-16, -14, -12, \ldots$ [*The difference between any term and the previous term is 2; $-10, -8, -6$.*]
2. $6, -2, -10, \ldots$ [*The difference between any term and the previous term is -8; $-18, -26, -34$.*]

Discuss with students the methods they used to determine the patterns.

1 Activating Prior Knowledge, Create Representations, Look for a Pattern, Debriefing This is an entry-level item. Students plot data and use their knowledge of linear functions to write a rule for $f(n)$. When debriefing, be sure that students understand that the domain of the function must be discrete because *n* represents the number of carbon atoms.

2–3 Debriefing Use Item 2 to assess whether students understand the meaning of subscripts. Also, note that finding that the differences are constant in Item 3 will help students attach meaning to the definition of common difference that follows.

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Universal Access

Understanding that a sequence represents a function may be difficult for students. To help with this concept, ask students to write a set of ordered pairs that shows the functional relationship between the term number and the terms in the sequence {4, 6, 8, 10, 12, 14}.

4–5 Look for a Pattern, Debriefing In Item 4, students must generalize, which is a difficult concept for some students. Direct students to the Math Tip. If students need additional guidance understanding that these expressions represent consecutive terms in the sequence, try asking these questions: What do a_{n+1} and a_n represent when $n = 1$? When $n = 2$? When $n = 3$?

Item 5 connects the concepts of sequence and common difference back to the opening context and shows that consecutive integers form an arithmetic sequence with $d = 1$. To extend this item, ask students whether it is possible to have an arithmetic sequence where $d = 0$. Any sequence where the terms remain constant satisfies this condition.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to arithmetic sequences. For the sequence in Item 7, each term is multiplied by 2 to find the next term; be sure students can verbalize why this pattern does not determine an arithmetic sequence.

Answers

6. arithmetic; 5 **7.** not arithmetic
8. 37, 46, 64

9–10 Activating Prior Knowledge, Create Representations, Look for a Pattern In Item 9, students use their ability to work with literal equations to solve the formula from Item 4 for a_{n+1}. In Item 10, students identify each term using math terminology. In order to more easily compare the recursive formula and the explicit formula, students need to understand that the expression $a_n = a_{n-1} + d$ is equivalent to the expression in Item 9. Elicit from students the fact that a_n is the term that follows a_{n-1}, just as a_{n+1} is the term that follows a_n.

Developing Math Language

Watch for students who interchange the terms *recursive formula* and *explicit formula*. As students respond to questions or discuss possible solutions to problems, monitor their use of these terms to ensure their understanding and ability to use language correctly and precisely.

My Notes

4. Use a_n and a_{n+1} to write a general expression for the common difference d.
$d = a_{n+1} - a_n$

5. Determine whether the numbers of carbon atoms in the first six alkanes {1, 2, 3, 4, 5, 6} form an arithmetic sequence. Explain why or why not.
Yes; The sequence is arithmetic because there is a common difference of 1 between each pair of consecutive terms.

Check Your Understanding

Determine whether each sequence is arithmetic. If the sequence is arithmetic, state the common difference.

6. 3, 8, 13, 18, 23, . . .

7. 1, 2, 4, 8, 16, . . .

8. Find the missing terms in the arithmetic sequence 19, 28, ______, ______, 55, ______.

MATH TIP

In a sequence, a_{n+1} is the term that follows a_n.

9. Write a formula for a_{n+1} in Item 4.
$a_{n+1} = a_n + d$

10. What information is needed to find a_{n+1} using this formula?
The value of the common difference and the value of the previous term are needed.

Finding the value of a_{n+1} in the formula you wrote in Item 9 requires knowing the value of the previous term. Such a formula is called a ***recursive formula***, which is used to determine a term of a sequence using one or more of the preceding terms.

The terms in an arithmetic sequence can also be written as the sum of the first term and a multiple of the common difference. Such a formula is called an ***explicit formula*** because it can be used to calculate any term in the sequence as long as the first term is known.

11. Complete the blanks for the sequence {4, 6, 8, 10, 12, 14, . . .} formed by the number of hydrogen atoms.

$a_1 = $ **4** $\quad d = $ **2**

$a_2 = 4 + $ **1** $\cdot 2 = 6$

$a_3 = 4 + $ **2** $\cdot 2 = 8$

$a_4 = 4 + $ **3** $\cdot 2 = $ **10**

$a_5 = 4 + $ **4** $\cdot 2 = $ **12**

$a_6 = 4 + $ **5** $\cdot 2 = $ **14**

$a_{10} = 4 + $ **9** $\cdot 2 = $ **22**

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Lesson 19-1
Arithmetic Sequences

My Notes

12. Write a general expression a_n in terms of n for finding the number of hydrogen atoms in an alkane molecule with n carbon atoms.
$a_n = 4 + (n - 1) \cdot 2$

13. Use the expression you wrote in Item 12 to find the number of hydrogen atoms in decane, the alkane with 10 carbon atoms. Show your work.
$a_{10} = 4 + (10 - 1) \cdot 2 = 4 + 9 \cdot 2 = 22$

14. Find the number of carbon atoms in a molecule of an alkane with 38 hydrogen atoms. $n = 18$

15. **Model with mathematics.** Use a_1, d, and n to write an explicit formula for a_n, the nth term of any arithmetic sequence.
$a_n = a_1 + (n - 1) \cdot d$

16. Use the formula from Item 15 to find the specified term in each arithmetic sequence.

a. Find the 40th term when $a_1 = 6$ and $d = 3$. 123

b. Find the 30th term of the arithmetic sequence 37, 33, 29, 25, ... −79

Example A

Hope is sending invitations for a party. The cost of the invitations is $5.00, and postage for each is $0.45. Write an expression for the cost of mailing the invitations in terms of the number of invitations mailed. Then calculate the cost of mailing 16 invitations.

Step 1: Identify a_1 and d.
The cost to mail the first invitation is equal to the cost of the invitations and the postage for that one invitation.
$a_1 = 5.00 + 0.45 = 5.45.$
The postage per invitation is the common difference, $d = 0.45$.

Step 2: Use the information from Step 1 to write a general expression for a_n.
If n equals the number of invitations mailed, then the expression for the cost of mailing n invitations is:
$a_n = a_1 + (n - 1)d$
$a_n = 5.45 + (n - 1)(0.45)$
$a_n = 5.45 + 0.45n - 0.45$
$a_n = 5.00 + 0.45n$

Step 3: Use the general expression to evaluate a_{16}.
The cost of mailing 16 invitations is found by solving for $n = 16$.
$a_{16} = 5.00 + 0.45(16) = 5.00 + 7.20 = 12.20.$

Try These A

Write an expression for the nth term of the arithmetic sequence, and then find the term.

a. Find the 50th term when $a_1 = 7$ and $d = -2$.
$a_n = 7 + (n - 1) \cdot -2; a_{50} = -91$

b. Find the 28th term of the arithmetic sequence 3, 7, 11, 15, 19, ...
$a_n = 3 + (n - 1) \cdot 4; a_{28} = 111$

c. Which term in the arithmetic sequence 15, 18, 21, 24, . . . is equal to 72?
$a_n = 15 + (n - 1) \cdot 3; n = 20$

ACTIVITY 19 Continued

11–14 Look for a Pattern, Create Representations, Debriefing Notice that Item 11 skips from finding the sixth term to the tenth term. This provides scaffolding to help students generalize in Item 12. Students who do not make the connection between Items 11 and 12 may try to use the fact that to get any term in the sequence, you add 2 to the previous term. However, this leads to a recursive definition: $a_1 = 4$, $a_n = a_{n-1} + 2$, while the item asks for the explicit definition in terms of n. Suggest to those students that they use their work in Item 11 to answer Item 12.

Item 13 gives students the opportunity to verify the formula from Item 12. If a student gives an incorrect answer, ask the student to compare the answer for the eleventh term in the sequence to the answer for the tenth term in Item 11 to see whether it seems reasonable.

Some students may simply write the eleventh term without using the formula. The purpose of this activity is to derive and use formulas. Ask any student who does not see the necessity for a formula to find the 500th term of the sequence.

15–16 Think-Pair-Share, Create Representations, Look for a Pattern, Debriefing Writing an explicit formula for a_n in Item 15 is the culmination of work done in Items 11 through 14. Have students share answers on whiteboards and verify that all students have correct responses. Then, in Item 16, students practice using the formula they derived.

Example A Think-Pair-Share, Debriefing Students can discuss how they determined the values of a_1 and d for this real-world scenario.

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ACTIVITY 19 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to use both explicit and recursive formulas to calculate the nth term of an arithmetic sequence. Ask students to think of situations in which one formula might be more useful than the other.

Answers

17. $a_n = 4 + (n - 1) \cdot 2 = 4 + (2n - 2) = 2n + 2 = f(n)$

18. -3.5

19. the 8th term

20. Shontelle can do this because $a_n = a_{n-1} + d$. The results are equivalent; both formulas give the same result.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 19-1 PRACTICE

21. not arithmetic

22. **a.** -3;
b. $a_n = 23 - 3n$
c. $a_n = a_{n-1} - 3$

23. **a.** $d = 4$
b. $a_n = 3 + 4(n - 1)$
c. $a_n = a_{n-1} + 3$

24. 13, 18, 23, 28, 33

25. $a_1 = \frac{9}{8}$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand basic concepts related to arithmetic sequences and finding the nth term of such a sequence. Be sure students understand that d can be negative, as in Item 22. Students may substitute incorrectly into the formulas. Encourage them to begin each problem by writing the general formula and identifying each value in the formula. For example, for Item 23b, students can write $a_n = a_1 + (n - 1)d$ followed by $a_1 = 3$, $d = 4$. Correctly identifying and substituting these values will be important in upcoming lessons.

My Notes

Check Your Understanding

17. Show that the expressions for a_n in Item 12 and $f(n)$ in Item 1 are equivalent.

18. Find the 14th term for the sequence defined below.

term	1	2	3	4
value	1.7	1.3	0.9	0.5

19. Determine which term in the sequence in Item 18 has the value -1.1.

20. **Express regularity in repeated reasoning.** Shontelle used both the explicit and recursive formulas to calculate the fourth term in a sequence where $a_1 = 7$ and $d = 5$. She wrote the following:

Explicit:
$a_n = a_1 + (n-1)d$
$a_4 = 7 + (4-1)5$
$a_4 = 7 + 3 \times 5$

Recursive:
$a_n = a_{n-1} + d$
$a_4 = a_3 + 5$
$a_4 = (a_2 + 5) + 5$
$a_4 = ((a_1 + 5) + 5) + 5$
$a_4 = ((7 + 5) + 5) + 5$

Explain why Shontelle can substitute $(a_2 + 5)$ for a_3 and $(a_1 + 5)$ for a_2. Compare the result that Shontelle found when using the recursive formula with the result of the explicit formula. What does this tell you about the formulas?

LESSON 19-1 PRACTICE

For Items 21–23, determine whether each sequence is arithmetic. If the sequence is arithmetic, then

a. state the common difference.
b. use the explicit formula to write a general expression for a_n in terms of n.
c. use the recursive formula to write a general expression for a_n in terms of a_{n-1}.

21. 1, 1, 2, 3, 5, 8, . . .

22. 20, 17, 14, 11, 8, . . .

23. 3, 7, 11, . . .

24. A sequence is defined by $a_1 = 13$, $a_n = 5 + a_{n-1}$. Write the first five terms in the sequence.

25. **Make sense of problems.** Find the first term.

n	3	4	5	6
a_n	$\frac{7}{8}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{1}{2}$

CONNECT TO HISTORY

Item 21 is a famous sequence known as the Fibonacci sequence. Find out more about this interesting sequence. You can find its pattern in beehives, pinecones, and flowers.

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Lesson 19-2
Arithmetic Series

ACTIVITY 19
continued

My Notes

Learning Targets:
- Write a formula for the *n*th partial sum of an arithmetic series.
- Calculate partial sums of an arithmetic series.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Create Representations

A ***series*** is the sum of the terms in a sequence. The sum of the first n terms of a series is the nth ***partial sum*** of the series and is denoted by S_n.

1. Consider the arithmetic sequence {4, 6, 8, 10, 12, 14, 16, 18}.

a. Find S_4.
$S_4 = 28$

b. Find S_5.
$S_5 = 40$

c. Find S_8.
$S_8 = 88$

d. How does $a_1 + a_8$ compare to $a_2 + a_7$, $a_3 + a_6$, and $a_4 + a_5$?
Each sum is 22.

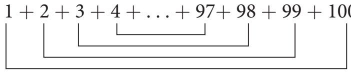
e. Make use of structure. Explain how to find S_8 using the value of $a_1 + a_8$.
Since there are 4 pairs of numbers with the same sum, $S_8 = 4(a_1 + a_8) = 4(22) = 88$.

2. Consider the arithmetic series $1 + 2 + 3 + \ldots + 98 + 99 + 100$.

a. How many terms are in this series?
100

b. If all the terms in this series are paired as shown below, how many pairs will there be?
50

$1 + 2 + 3 + 4 + \ldots + 97 + 98 + 99 + 100$



c. What is the sum of each pair of numbers?
101

d. Construct viable arguments. Find the sum of the series. Explain how you arrived at the sum.
There are 50 pairs of numbers, each with a sum of 101, so the sum is 50(101) = 5050.

CONNECT TO HISTORY

A story is often told that in the 1780s, a German schoolmaster decided to keep his students quiet by having them find the sum of the first 100 integers. One young pupil was able to name the sum immediately. This young man, Carl Friedrich Gauss, would become one of the world's most famous mathematicians. He reportedly used the method in Item 2 to find the sum, using mental math.

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Lesson 19-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–4 #5–6
Example A
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to determine the sum of the first 4 terms of each sequence.

1. 1, 4, 7, 10, … [22]
2. 3, 5, 7, 9, 11, … [24]
3. 2, 9, 16, 23, 30, 37, … [50]

Discuss the methods students used to find the sums.

Developing Math Language

As you guide students through their learning of the terms *series* and *partial sum*, explain meanings in terms that are accessible for your students. For example, point out that the word *part* is included in the word *partial*. Use the concrete examples in the items to help students gain understanding. Encourage students to make notes about these terms and their understanding of what they mean and how to use them to describe precise mathematical concepts and processes.

1–2 Look for a Pattern These items provide the scaffolding for students to develop the formula for the partial sum of an arithmetic series, which students will do in Item 3.

3–4 Think-Pair-Share In Item 3, students develop the general formula $s_n = \frac{n}{2}(a_1 + a_n)$ for the partial sum of an arithmetic series.

Some students may be concerned about the case when n is odd. If $n = 5$, students can consider this as representing $\frac{5}{2}$, or $2\frac{1}{2}$, pairs of terms. The expression $\frac{n}{2}$ represents the middle term in the series, which will be equal to half the sum of the first and last terms. Thus the formula holds for the case when n is odd.

It is important that students share their answers to Item 3 to verify that all students understand how to derive the formula.

In Item 4, students verify that the formula they just found gives the correct sums, the same ones they found in Item 1.

ELL Support

Students may use the term *series* incorrectly because it sounds like a plural word and because it is sometimes used in everyday language to refer to a sequence. In fact, a common error is to use *series* interchangeably with *sequence*. Monitor students' use of *series* carefully to be sure they understand that the term refers to a single sum.

My Notes

3. Consider the arithmetic series $a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n$.

$$a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n$$

a. Write an expression for the number of pairs of terms in this series.

$\frac{n}{2}$

b. Write a formula for S_n, the partial sum of the arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$

4. Use the formula from Item 3b to find each partial sum of the arithmetic sequence {4, 6, 8, 10, 12, 14, 16, 18}. Compare your results to your answers in Item 1.

Results in Items 4a–c should be the same as 1a–c.

a. S_4

$S_4 = \frac{4}{2}(4 + 10) = 2(14) = 28$

b. S_5

$S_5 = \frac{5}{2}(4 + 12) = \frac{5}{2}(16) = 40$

c. S_8

$S_4 = \frac{8}{2}(4 + 18) = 4(22) = 88$

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Lesson 19-2
Arithmetic Series

5. A second form of the formula for finding the partial sum of an arithmetic series is $S_n = \frac{n}{2}[2a_1 + (n-1)d]$. Derive this formula, starting with the formula from Item 3b of this lesson and the nth term formula, $a_n = a_1 + (n-1)d$, from Item 15 of the previous lesson.

$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[a_1 + (a_1 + (n-1) \cdot d)] = \frac{n}{2}[2a_1 + (n-1) \cdot d]$

6. Use the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ to find the indicated partial sum of each arithmetic series. Show your work.

a. $3 + 8 + 13 + 18 + \ldots; S_{20}$

$S_{20} = \frac{20}{2}[2 \cdot 3 + (20-1) \cdot 5] = 10(6 + 19 \cdot 5) = 10(101) = 1010$

b. $-2 - 4 - 6 - 8 - \ldots; S_{18}$

$S_{18} = \frac{18}{2}[2(-2) + (18-1)(-2)] = 9[-4 + 17(-2)] = 9(-38) = -342$

Example A

Find the partial sum S_{10} of the arithmetic series with $a_1 = -3$, $d = 4$.

Step 1: Find a_{10}.
The terms are $-3, 1, 5, 9, \ldots$.
$a_1 = -3$
$a_{10} = a_1 + (n-1)d = -3 + (10-1)(4) = -3 + (9)(4) = -3 + 36 = 33$

Step 2: Substitute for n, a_1, and a_{10} in the formula. Simplify.
$S_{10} = \frac{n}{2}(a_1 + a_n) = \frac{10}{2}(-3 + 33) = 5(30) = 150$
Or use the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$:
$S_{10} = \frac{10}{2}[2(-3) + (10-1)4] = 5[-6 + 36] = 150$

Try These A

Find the indicated sum of each arithmetic series. Show your work.

a. Find S_8 for the arithmetic series with $a_1 = 5$ and $a_8 = 40$.
180

b. $12 + 18 + 24 + 30 + \ldots; S_{10}$
390

c. $30 + 20 + 10 + 0 + \ldots; S_{25}$
-2250

My Notes

5–6 Think-Pair-Share, Create Representations, Look for a Pattern In Item 5, students show how to derive an alternate formula for the partial sum of an arithmetic series, $S_n = \frac{n}{2}[2a_1 + (n-1)d]$. Point out to students that they will use one or the other formula, depending upon which values they are given for a particular series. To use the formula from Item 3, they need to know n, a_1, and a_n. For this alternate formula, they need n, a_1, and d.

In Item 6, students apply the alternate formula to find partial sums.

Example A Think-Pair-Share, Debriefing Students can discuss the steps they took to determine the solution. They should explain which of the two partial sum formulas they chose for each item in the Try These, and why.

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Check Your Understanding

Debrief students' answers to these items to ensure that they understand the concepts related to finding partial sums of arithmetic series. Ask students whether they could find the sum of all terms in an arithmetic sequence.

Answers

7. $\frac{n}{2} = 3$, $n = 6$; $a_1 = 12$; $a_6 = 37$
8. 12, 17, 22, 27, 32, 37; $12 + 17 + 22 + 27 + 32 + 37 = 147$
9. Sample answer: Use $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ when a_n is unknown.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 19-2 PRACTICE

10. $S_{10} = 265$
11. $S_{12} = 180$
12. $S_{10} = 72$
13. 180 seats

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand basic concepts related to finding the nth partial sum of an arithmetic series. The goal of this lesson is for students to develop and use the formulas; watch for students who find sums by simply adding terms in the series. Insist that they show their work using the formulas.

My Notes

Check Your Understanding

7. Explain what each term of the equation $S_6 = 3(12 + 37) = 147$ means in terms of n and a_n.
8. Find each term of the arithmetic series in Item 7, and then verify the given sum.
9. When would the formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$ be preferred to the formula $S_n = \frac{n}{2}(a_1 + a_n)$?

LESSON 19-2 PRACTICE

10. Find the partial sum S_{10} of the arithmetic series with $a_1 = 4$, $d = 5$.
11. Find the partial sum S_{12} of the arithmetic series $26 + 24 + 22 + 20 + \ldots$.
12. Find the sum of the first 10 terms of an arithmetic sequence with an eighth term of 8.2 and a common difference of 0.4.
13. **Model with mathematics.** An auditorium has 12 seats in the first row, 15 in the second row, and 18 in the third row. If this pattern continues, what is the total number of seats for the first eight rows?

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Lesson 19-3
Sigma Notation

My Notes

Learning Targets:

- Identify the index, lower and upper limits, and general term in sigma notation.
- Express the sum of a series using sigma notation.
- Find the sum of a series written in sigma notation.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Create Representations

In the Binomial Theorem activity in Unit 3, you were introduced to a shorthand notation called ***sigma notation*** (Σ). It is used to express the sum of a series.

The expression $\sum_{n=1}^{4}(2n+5)$ is read "the sum from $n = 1$ to $n = 4$ of $2n + 5$."

To expand the series to show the terms of the series, substitute 1, 2, 3, and 4 into the expression for the general term. To find the sum of the series, add the terms.

$$\sum_{n=1}^{4}(2n+5) = (2 \cdot 1 + 5) + (2 \cdot 2 + 5) + (2 \cdot 3 + 5) + (2 \cdot 4 + 5)$$
$$= 7 + 9 + 11 + 13 = 40$$

MATH TIP

upper limit of summation

$\sum_{n=1}^{4}(2n+5)$ ← general term

index of summation; lower limit of summation

MATH TIP

To find the first term in a series written in sigma notation, substitute the value of the lower limit into the expression for the general term.

To find subsequent terms, substitute consecutive integers that follow the lower limit, stopping at the upper limit.

Example A

Evaluate $\sum_{j=1}^{6}(2j-3)$.

Step 1: The values of j are 1, 2, 3, 4, 5, and 6. Write a sum with six addends, one for each value of the variable.
$= [2(1) - 3] + [2(2) - 3] + [2(3) - 3] + [2(4) - 3] +$
$[2(5) - 3] + [2(6) - 3]$

Step 2: Evaluate each expression.
$= -1 + 1 + 3 + 5 + 7 + 9$

Step 3: Simplify.
$= 24$

Try These A

a. **Use appropriate tools strategically.** Write the terms in the series $\sum_{n=1}^{8}(3n-2)$. Then find the indicated sum.

$1 + 4 + 7 + 10 + 13 + 16 + 19 + 22 = 92$

b. Write the sum of the first 10 terms of $80 + 75 + 70 + 65 + \ldots$ using sigma notation.

Sample answer: $\sum_{n=1}^{10}(85-5n)$

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Lesson 19-3

PLAN

Pacing: 1 class period

Chunking the Lesson
Example A
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to use one of the formulas for S_n from Lesson 19-2 to find the indicated partial sum of each arithmetic sequence.

1. 2, 4, 6, 8, …; S_4 *[20]*
2. $-16, -6, 4, 14, 24, 34, \ldots; S_6$ *[54]*
3. 5, 12, 19, 26, 31, …; S_5 *[90]*

Discuss the formulas students used to determine their answers.

TEACHER TO TEACHER

Students were introduced to summation notation, or sigma notation, in Unit 3. Note that if the lower limit is 1, the upper limit is equal to the number of terms in the series. If the lower limit is an integer other than 1, say i, and the upper limit is j, then the number of terms in the series will be $j - i + 1$.

Example A Marking the Text, Simplify the Problem, Think-Pair-Share, Debriefing You might start by finding the value of $2j - 3$ when $j = 1$ and $j = 2$. Now have students work in small groups to find the values of $2j - 3$ for $j = 3$ through $j = 6$. Elicit the fact from students that they now must add all of these values because sigma notation represents their sum.

Differentiating Instruction

Support students who struggle with sigma notation by providing this additional practice.

Expand the series and find the sum.

1. $\sum_{n=1}^{5}(3n+1)$ *[50]*
2. $\sum_{n=1}^{8}(5-2n)$ *[−32]*
3. $\sum_{n=1}^{10}2n$ *[110]*

Then assign the following three items, in which the lower limit is not 1 and the upper limit is not the number of terms in the sequence.

Expand the series and find the sum.

4. $\sum_{n=0}^{50}(3n+1)$ *[3876]*
5. $\sum_{n=5}^{80}(5-2n)$ *[−6080]*
6. $\sum_{n=100}^{1000}2n$ *[991,100]*

ACTIVITY 19 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they know and understand how to calculate d, a_n, and S_n. Ask them to explain the meaning of each term in their formulas.

Answers

1. $d = a_{n+1} - a_n$
2. $a_n = a_1 + (n - 1) \cdot d$
3. $S_n = \frac{n}{2}(a_1 + a_n)$ or

   $S_n = \frac{n}{2}[2a_1 + (n-1)d]$

Technology Tip

Many graphing calculators and computer algebra systems can calculate sums like the ones in this lesson. Typically, Sum or Sequence functions return the sum when the general term, index of summation, and upper and lower limits are entered; consult your manual for specific instructions. You may wish to allow students to check their answers using technology until they gain confidence working with sigma notation.

For additional technology resources, visit SpringBoard Digital.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 19-3 PRACTICE

4. $S_{15} = 345$
5. $S_{20} = 440$
6. $S_{10} = 135$
7. index: n; upper limit: 15; lower limit: 1; general term: $3n - 1$
8. Sample answer: $\sum_{n=1}^{7} -1 + 4n$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to find a sum written in sigma notation. If students need more practice, find arithmetic sequences from the previous lessons in this activity and have students use sigma notation to write partial sums for these sequences. (Students may choose which partial sums, or you may wish to assign specific partial sums.) Students can then exchange papers and find each indicated sum.

My Notes

Check Your Understanding

Summarize the following formulas for an arithmetic series.

1. common difference $d =$ ______
2. nth term $a_n =$ ______
3. sum of first n terms $S_n =$ ______ or $S_n =$ ______

LESSON 19-3 PRACTICE

Find the indicated partial sum of each arithmetic series.

4. $\sum_{n=1}^{15}(3n - 1)$
5. $\sum_{k=1}^{20}(2k + 1)$
6. $\sum_{j=5}^{10} 3j$
7. Identify the index, upper and lower limits, and general term of Item 4.
8. **Attend to precision.** Express the following sum using sigma notation: $3 + 7 + 11 + 15 + 19 + 23 + 27$

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ACTIVITY 19 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 19-1

1. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, state the common difference.
a. 4, 5, 7, 10, . . .
b. 5, 7, 9, 11, . . .
c. 12, 9, 6, 3, . . .

2. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, use the explicit formula to write a general expression for a_n in terms of n.
a. 4, 12, 20, 28, . . .
b. 5, 10, 20, 40, . . .
c. 4, 0, −4, −8, . . .

3. Determine whether or not each sequence is arithmetic. If the sequence is arithmetic, use the recursive formula to write a general expression for a_n in terms of a_{n-1}.
a. 7, 7.5, 8, 8.5, . . .
b. 6, 7, 8, 9, . . .
c. −2, 4, −8, . . .

4. Find the indicated term of each arithmetic sequence.
a. $a_1 = 4$, $d = 5$; a_{15}
b. 14, 18, 22, 26, . . .; a_{20}
c. 45, 41, 37, 33, . . .; a_{18}

5. Find the sequence for which a_8 does NOT equal 24.
A. 3, 6, 9, . . .
B. −32, −24, −16, . . .
C. 108, 96, 84, . . .
D. −8, −4, 0, . . .

6. A radio station offers a $100 prize on the first day of a contest. Each day that the prize money is not awarded, $50 is added to the prize amount. If a contestant wins on the 17th day of the contest, how much money will be awarded?

7. If $a_4 = 20$ and $a_{12} = 68$, find a_1, a_2, and a_3.

8. Find the indicated term of each arithmetic sequence.
a. $a_1 = -2$, $d = 4$; a_{12}
b. 15, 19, 23, 27, . . .; a_{10}
c. 46, 40, 34, 28, . . .; a_{20}

9. What is the first value of n that corresponds to a positive value? Explain how you found your answer.

n	1	2	3	4	5
a_n	−42.5	−37.8	−33.1	−28.4	−23.7

10. Find the first four terms of the sequence with $a_1 = \frac{2}{3}$ and $a_n = a_{n-1} + \frac{1}{6}$.

11. If $a_1 = 3.1$ and $a_5 = -33.7$, write an expression for the sequence and find a_2, a_3, and a_4.

Lesson 19-2

12. Find the indicated partial sum of each arithmetic series.
a. $a_1 = 4$, $d = 5$; S_{10}
b. 14 + 18 + 22 + 26 + . . .; S_{12}
c. 45 + 41 + 37 + 33 + . . .; S_{18}

13. Find the indicated partial sum of each arithmetic series.
a. 1 + 3 + 5 + . . .; S_6
b. 1 + 3 + 5 + . . .; S_{10}
c. 1 + 3 + 5 + . . .; S_{12}
d. Explain the relationship between n and S_n in parts a–c.

14. Find the indicated partial sum of the arithmetic series.

$0 + (x + 2) + (2x + 4) + (3x + 6) + \dots; S_{10}$

A. $9x + 18$
B. $10x + 20$
C. $45x + 90$
D. $55x + 110$

15. Two companies offer you a job. Company A offers you a $40,000 first-year salary with an annual raise of $1500. Company B offers you a $38,500 first-year salary with an annual raise of $2000.
a. What would your salary be with Company A as you begin your sixth year?
b. What would your salary be with Company B as you begin your sixth year?
c. What would be your total earnings with Company A after 5 years?
d. What would be your total earnings with Company B after 5 years?

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ACTIVITY 19 Continued

ACTIVITY PRACTICE

1. a. no
b. yes; $d = 2$
c. yes; $d = -3$
2. a. yes; $a_n = -4 + 8n$
b. no
c. yes; $a_n = 8 - 4n$
3. a. yes; $a_n = a_{n-1} + 0.5$
b. yes; $a_n = a_{n-1} + 1$
c. no
4. a. $a_{15} = 74$
b. $a_{20} = 90$
c. $a_{18} = -23$
5. D
6. $900
7. $a_1 = 2$, $a_2 = 8$, and $a_3 = 14$
8. a. $a_{12} = 42$
b. $a_{10} = 51$
c. $a_{20} = -68$
9. $n = 11$. Sample explanation: I divided −23.7 by the common difference of 4.7; the result is between 5 and 6, so it will take 6 more terms to get a positive value.
10. $\frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}$
11. $a_n = -3.1 + (n - 1)9.2$; $a_2 = -6.1$, $a_3 = -15.3$, $a_4 = -24.5$
12. a. $S_{10} = 265$
b. $S_{12} = 432$
c. $S_{18} = 198$
13. a. $S_6 = 36$
b. $S_{10} = 100$
c. $S_{12} = 144$
d. $S_n = n^2$
14. C
15. a. $47,500
b. $48,500
c. $215,000
d. $212,500

16. $d = 4$
17. $d = -10$; $S_7 = 119$
18. $a_1 = 6.24$; $S_9 = 70.56$
19. a. 9 prizes
b. $2700
20. B
21. $S = 75(1 + 150) = 11{,}325$
22. $S_n = \frac{n}{2}(a_1 + a_5) = \frac{5}{2}(20 + 12) = 80$
23. a. $\sum_{j=1}^{5}(5 - 6j) = -65$
b. $\sum_{j=1}^{20} 5j = 1050$
c. $\sum_{j=5}^{15}(5 - j) = -55$
24. yes; $\sum_{j=1}^{10}(2j + 1) = 120$;
$\sum_{j=1}^{5}(2j + 1) = 35$;
$\sum_{j=6}^{10}(2j + 1) = 85$; $120 = 35 + 85$
25. yes; $\sum_{j=4}^{9}(j - 7) = -3$;
$\sum_{j=1}^{9}(j - 7) = -18$;
$\sum_{j=1}^{3}(j - 7) = -15$;
$-3 = -18 - (-15)$
26. B
27. a. $\sum_{j=1}^{6}(j + 3) = 39$
b. $\sum_{j=10}^{15}(j - 12) = 3$
c. $\sum_{j=1}^{8}(4j) = 144$
28. $\sum_{j=4}^{8}(-3j + 29) = 55$ and
$\sum_{j=4}^{8} -3j + 29 = -61$;
$\sum_{j=4}^{8}(-3j + 29)$ is greater.
29. D
30. $\sum_{j=1}^{5}\left(\frac{j \cdot \pi}{2}\right) = \frac{15\pi}{2}$
31. Sample answer: Because there is a constant difference between sequential terms of a sequence, terms can be paired to represent a constant sum. Therefore, the partial sum is the product of the number of pairs times the sum.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

16. If $S_{12} = 744$ and $a_1 = 40$, find d.

17. In an arithmetic series, $a_1 = 47$ and $a_7 = -13$, find d and S_7.

18. In an arithmetic series, $a_9 = 9.44$ and $d = 0.4$, find a_1 and S_9.

19. The first prize in a contest is $500, the second prize is $450, the third prize is $400, and so on.
a. How many prizes will be awarded if the last prize is $100?
b. How much money will be given out as prize money?

20. Find the sum of $13 + 25 + 37 + \ldots + 193$.
A. 1339
B. 1648
C. 1930
D. 2060

21. Find the sum of the first 150 natural numbers.

22. A store puts boxes of canned goods into a stacked display. There are 20 boxes in the bottom layer. Each layer has two fewer boxes than the layer below it. There are five layers of boxes. How many boxes are in the display? Explain your answer.

Lesson 19-3

23. Find the indicated partial sum of each arithmetic series.
a. $\sum_{j=1}^{5}(5 - 6j)$
b. $\sum_{j=1}^{20} 5j$
c. $\sum_{j=5}^{15}(5 - j)$

24. Does $\sum_{j=1}^{10}(2j + 1) = \sum_{j=1}^{5}(2j + 1) + \sum_{j=6}^{10}(2j + 1)$? Verify your answer.

25. Does $\sum_{j=4}^{9}(j - 7) = \sum_{j=1}^{9}(j - 7) - \sum_{j=1}^{3}(j - 7)$? Verify your answer.

26. Which statement is true for the partial sum $\sum_{j=1}^{n}(4j + 3)$?
A. For $n = 5$, the sum is 35.
B. For $n = 7$, the sum is 133.
C. For $n = 10$, the sum is 230.
D. For $n = 12$, the sum is 408.

27. Evaluate.
a. $\sum_{j=1}^{6}(j + 3)$
b. $\sum_{j=10}^{15}(j - 12)$
c. $\sum_{j=1}^{8}(4j)$

28. Which is greater: $\sum_{j=4}^{8}(-3j + 29)$ or $\sum_{j=4}^{8} -3j + 29$?

29. Which expression is the sum of the series $7 + 10 + 13 + \ldots + 25$?
A. $\sum_{j=1}^{7} 4 + 3j$
B. $\sum_{j=1}^{7}(4 - 3j)$
C. $\sum_{j=1}^{7}(3 + 4j)$
D. $\sum_{j=1}^{7}(4 + 3j)$

30. Evaluate $\sum_{j=1}^{5}\left(\frac{j \bullet \pi}{2}\right)$.

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

31. How does the common difference of an arithmetic sequence relate to finding the partial sum of an arithmetic series?

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Geometric Sequences and Series

ACTIVITY 20

Squares with Patterns

Lesson 20-1 Geometric Sequences

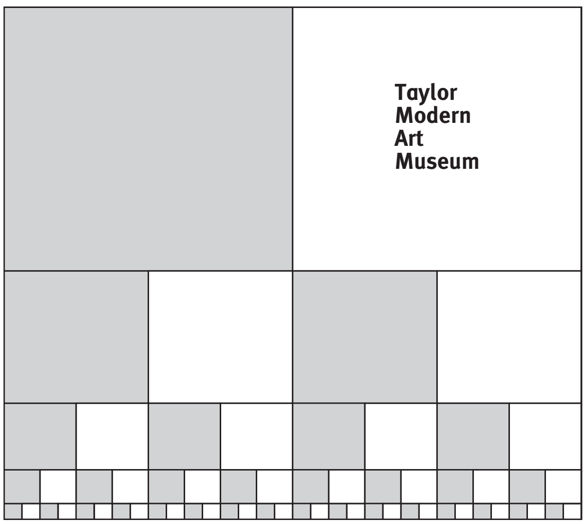
My Notes

Learning Targets:

- Determine whether a given sequence is geometric.
- Find the common ratio of a geometric sequence.
- Write an expression for a geometric sequence, and calculate the *n*th term.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations

Meredith is designing a mural for an outside wall of a warehouse that is being converted into the Taylor Modern Art Museum. The mural is 32 feet wide by 31 feet high. The design consists of squares in five different sizes that are painted black or white as shown below.



1. Let Square 1 be the largest size and Square 5 be the smallest size. For each size, record the length of the side, the number of squares of that size in the design, and the area of the square.

Square #	Side of Square (ft)	Number of Squares	Area of Square (ft^2)
1	16	2	256
2	8	4	64
3	4	8	16
4	2	16	4
5	1	32	1

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Common Core State Standards for Activity 20

HSA-SSE.B.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.★

HSF-BF.A.1 Write a function that describes a relationship between two quantities.★

HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.

HSF-BF.A.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

ACTIVITY 20

Guided

Activity Standards Focus

In Activity 20, students learn about geometric sequences and series. First they learn to identify and define a geometric sequence, including identifying the common ratio. Then they will examine and find sums of finite and infinite geometric series. Students will use information they learned in the previous activity about writing sequences and series in explicit and recursive forms.

Lesson 20-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2–3 #4–5
Check Your Understanding
#9 #10–12
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find the next number in each sequence.

1. 1, 4, 7, 10, … [*13*]

2. $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ $\left[\frac{5}{2}\right]$

3. −50, 40, −30, 20, … [−*10*]

1 Create Representations, Debriefing The purpose of this item is to generate several sequences of numbers that will be used throughout the activity. If students have difficulty completing the table, refer them to information in the opening paragraph about the overall dimensions of the mural. Using the dimensions along with the visual representations will allow students to determine the information needed to complete the table. Debrief after this item to be certain all students have the correct answers before they move on to the next items.

Differentiating Instruction

Support students in completing Item 1 by having them label squares in the diagram as 1, 2, 3, 4, 5. This will help them complete the table without having to reread the problem statement several times.

2–3 Look for a Pattern, Vocabulary Organizer, Quickwrite, Debriefing
Students may notice the multiplicative patterns in the last three columns of the table. They should be able to identify the first column as an arithmetic sequence with a common difference of 1.

Use Item 3 to assess student understanding of a geometric sequence and a common ratio. Be sure that students understand that the common ratio is the number multiplied by the previous item to generate the next term. So, while students may say that each term in the side of squares sequence is divided by 2 to generate the next term, the common ratio is in fact $\frac{1}{2}$.

Developing Math Language

Geometric sequence and *common ratio* are defined for the student. The Math Tip may further help with the understanding of these new concepts. As you guide students through their learning of these terms, explain meanings in terms that are accessible for your students. Students should add the new terms to their math notebooks, including notes about their meanings and how to use them to describe precise mathematical concepts and processes.

Universal Access

Watch for students who write the reciprocals of the common ratios for Item 3b. Remind students of the definition of a common ratio—it is the ratio of *consecutive* terms. This means the next term is the numerator and the previous term is the denominator. Identifying *r* correctly will be important when students study geometric series in the upcoming lessons.

2. Refer to the table in Item 1.

a. Describe any patterns that you notice in the table.

Sample answer: As the square number increases by 1, the side length of a square is divided by 2. The number of squares doubles each time the square number increases by 1. The areas are perfect squares. Each product of the side of a square and the number of squares is 32.

b. Each column of numbers forms a sequence of numbers. List the four sequences that you see in the columns of the table.

1, 2, 3, 4, 5; 16, 8, 4, 2, 1; 2, 4, 8, 16, 32; 256, 64, 16, 4, 1

c. Are any of those sequences arithmetic? Why or why not?

The only sequence that is arithmetic is 1, 2, 3, 4, 5, because it is the only sequence with a common difference. The common difference is 1.

MATH TIP

To find the common difference in an arithmetic sequence, subtract the preceding term from the following term.

To find the common ratio in a geometric sequence, divide any term by the preceding term.

A ***geometric sequence*** is a sequence in which the ratio of consecutive terms is a constant. The constant is called the ***common ratio*** and is denoted by *r*.

3. Consider the sequences in Item 2b.

a. List those sequences that are geometric.

Side of square: 16, 8, 4, 2, 1

Number of squares: 2, 4, 8, 16, 32

Area of square: 256, 64, 16, 4, 1

b. State the common ratio for each geometric sequence.

Side of square: $r = \frac{1}{2}$

Number of squares: $r = 2$

Area of square: $r = \frac{1}{4}$

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Lesson 20-1
Geometric Sequences

My Notes

4. Use a_n and a_{n-1} to write a general expression for the common ratio r.

$r = \frac{a_n}{a_{n-1}}$

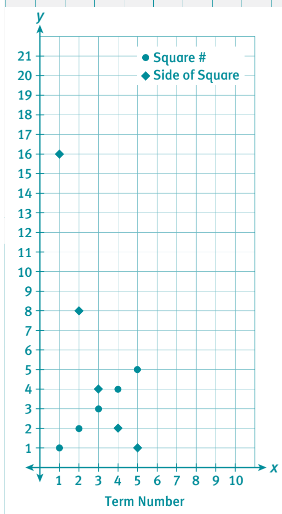
5. Consider the sequences in the columns of the table in Item 1 that are labeled Square # and Side of Square.

a. Plot the Square # sequence by plotting the ordered pairs (term number, square number).

b. Using another color or symbol, plot the Side of Square sequence by plotting the ordered pairs (term number, side of square).

c. Is either sequence a linear function? Explain why or why not.

The Square # plot is linear because there is a constant rate of change. Each time the term number increases by 1, the Square # increases by 1. The Side of Square sequence does not increase by a constant rate of change. Each time the term number increases by 1, the Side of Square decreases by an increasingly smaller amount.



Check Your Understanding

6. Determine whether each sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, state the common difference. If it is geometric, state the common ratio.

a. 3, 9, 27, 81, 243, . . .

b. 1, −2, 4, −8, 16, . . .

c. 4, 9, 16, 25, 36, . . .

d. 25, 20, 15, 10, 5, . . .

7. Use a_{n+1} and a_{n+2} to write an expression for the common ratio r.

8. Describe the graph of the first 5 terms of a geometric sequence with the first term 2 and the common ratio equal to 1.

9. Reason abstractly. Use the expression from Item 4 to write a *recursive formula* for the term a_n and describe what the formula means.

$a_n = a_{n-1}r$; We can find a term in a geometric sequence by knowing the previous term and the common ratio.

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4–5 Think-Pair-Share, Create Representations, Quickwrite, Debriefing Writing a general expression for the common difference in an arithmetic sequence, which students did in the preceding activity, should help them respond to Item 4. Be sure that they write the expression for r as $\frac{a_{n+1}}{a_n}$. Other variations may exist, but this is the most commonly used expression.

Item 5 allows students to look at an arithmetic sequence and a geometric sequence graphically. It can be used to reinforce the concept that an arithmetic sequence is a linear function in which the domain is a set of positive, consecutive integers. Students should be able to relate the common difference of the sequence to the constant rate of change of a linear function. Students should also note that the terms in the geometric sequence are not linear because the constant related to a geometric sequence is multiplicative rather than additive.

Check Your Understanding
Debrief students' answers to these items to ensure that they can identify arithmetic and geometric sequences. Ask students to describe their methods for identifying these sequences, and, if time allows, present additional examples of sequences and have students identify them using their described methods.

Answers

6. a. geometric; $r = 3$

b. geometric; $r = -2$

c. neither

d. arithmetic; $d = -5$

7. $r = \frac{a_{n+2}}{a_{n+1}}$

8. The graph will consist of the points (1, 2), (2, 2), (3, 2), (4, 2), and (5, 2).

9 Create Representations If necessary, review the meaning of *recursive formula*. Ask students to describe how they used their answer to Item 4 to write the recursive formula.

10–12 Think-Pair-Share, Look for a Pattern, Create Representations, Debriefing Notice that Item 10 skips from finding the sixth term in the sequence to finding the tenth term. This provides scaffolding to help students generalize in Item 11.

Item 11 is a culminating item for the first part of the activity. Have students share answers on whiteboards and verify that all students have correct responses.

In Item 12, students practice using the formula derived in Item 11. The purpose of this activity is to derive and use formulas. Be sure that students do not use repeated multiplication to find the missing terms. Students should show how to use the formula to find the desired term. They may verify the results with repeated multiplication if they choose.

Technology Tip

Students can use a calculator to find the terms of a geometric sequence.

Enter the first term and press ENTER.

Press x and enter the common ratio.

Press ENTER repeatedly to find each subsequent term.

For some students, writing this process in their notes will be helpful, as they can refer to it again as they work through the course.

For additional technology resources, visit SpringBoard Digital.

Differentiating Instruction

Support students who struggle with Item 10 by helping them transition from recognizing a repeated multiplication pattern to being able to express the pattern using exponents. Have students write each term in Item 10 in the form below.

$a_2 = 4 \cdot 2 = 4 \cdot 2^{\square} = 8$

$a_3 = 4 \cdot 2 \cdot \square = 4 \cdot 2^{\square} = 16$

$a_4 = 4 \cdot 2 \cdot \square \cdot \square = 4 \cdot 2^{\square} = 32$

My Notes

The terms in a geometric sequence also can be written as the product of the first term and a power of the common ratio.

10. For the geometric sequence {4, 8, 16, 32, 64, ... }, identify a_1 and r. Then fill in the missing exponents and blanks.

$a_1 =$ **4** $\qquad r =$ **2**

$a_2 = 4 \cdot 2^{1} = 8$

$a_3 = 4 \cdot 2^{2} = 16$

$a_4 = 4 \cdot 2^{3} =$ **32**

$a_5 = 4 \cdot 2^{4} =$ **64**

$a_6 = 4 \cdot 2^{5} =$ **128**

$a_{10} = 4 \cdot 2^{9} =$ **2048**

11. Use a_1, r, and n to write an *explicit formula* for the nth term of any geometric sequence.

$a_n = a_1 \cdot r^{n-1}$

12. Use the formula from Item 11 to find the indicated term in each geometric sequence.

a. 1, 2, 4, 8, 16, . . . ; a_{16}

32,768

b. 4096, 1024, 256, 64, . . . ; a_9

$\frac{1}{16}$ **or 0.0625**

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Lesson 20-1
Geometric Sequences

ACTIVITY 20
continued

My Notes

Check Your Understanding

13. a. Complete the table for the terms in the sequence with $a_1 = 3$; $r = 2$.

Term	Recursive $a_n = a_{n-1} \cdot r$	Explicit $a_n = a_1 \cdot r^{n-1}$	Value of Term
a_1	3	$3 \cdot 2^{1-1} = 3$	3
a_2	$3 \cdot 2$	$3 \cdot 2^{2-1} = 3 \cdot 2$	6
a_3	$(3 \cdot 2) \cdot 2$	$3 \cdot 2^{3-1} = 3 \cdot 2^2$	12
a_4	**$((3 \cdot 2) \cdot 2) \cdot 2$**	**$3 \cdot 2^{4-1} = 3 \cdot 2^3$**	**24**
a_5	**$(((3 \cdot 2) \cdot 2) \cdot 2) \cdot 2$**	**$3 \cdot 2^{5-1} = 3 \cdot 2^4$**	**48**

b. What does the product $(3 \cdot 2)$ represent in the recursive expression for a_3?

c. Express regularity in repeated reasoning. Compare the recursive and explicit expressions for each term. What do you notice?

LESSON 20-1 PRACTICE

14. Write a formula that will produce the sequence that appears on the calculator screen below.

```
5*3
                15
Ans*3
                45
               135
               405
```

15. Determine whether each sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, state the common difference, and if it is geometric, state the common ratio.

a. 3, 5, 7, 9, 11, . . .

b. 5, 15, 45, 135, . . .

c. $6, -4, \frac{8}{3}, -\frac{16}{9}, \ldots$

d. 1, 2, 4, 7, 11, . . .

16. Find the indicated term of each geometric sequence.

a. $a_1 = -2$, $r = 3$; a_8

b. $a_1 = 1024$, $r = -\frac{1}{2}$; a_{12}

17. Attend to precision. Given the data in the table below, write both a recursive formula and an explicit formula for a_n.

n	1	2	3	4
a_n	0.25	0.75	2.25	6.75

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ACTIVITY 20 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the difference between the explicit form and the recursive form of a geometric sequence. Be sure students understand and can verbalize the meaning of every term and variable in the formulas.

Answers

13. a. See student page.

b. $a_1 \cdot r$

c. Sample answer: The number of times that 2 is a factor in the recursive expression matches the value of the exponent in the explicit formula.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 20-1 PRACTICE

14. $a_n = 5(3)^{n-1}$

15. a. arithmetic; $d = 2$

b. geometric; $r = 3$

c. geometric; $r = -\frac{2}{3}$

d. neither

16. a. $a_8 = -4374$

b. $a_{12} = -0.5$

17. recursive: $a_{n+1} = a_n \cdot 3$;
explicit: $a_n = 0.25(3)^{n-1}$

ADAPT

Check students' answers to the Lesson Practice to ensure that they can differentiate between an arithmetic and a geometric sequence. Students may benefit from making a table or other graphic organizer that compares and contrasts the two types of sequences and includes examples of each. Encourage students to refer to their graphic organizers as needed.

Lesson 20-2

PLAN

Materials

- paper squares
- scissors

Pacing: 1 class period

Chunking the Lesson

#1 Example A

Check Your Understanding

#4a #4b–d #5–6

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find the eighth term in each geometric sequence.

1. 4, 16, 64, 256, … [*65,536*]

2. 1, −3, 9, −27, … [*−16,384*]

3. 4, 2, 1, 0.5, … $\left[\frac{1}{32}\right]$

1 Close Reading, Think-Pair-Share Look for groups of students who are able to follow the steps to derive the formula successfully, and have them share their work with the class.

TEACHER to TEACHER

Students focus on a formula for determining the sum of a finite geometric series: $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$, $r \neq 1$. They follow the steps given to derive the formula and then apply it. A common student error when applying the formula is to use $n - 1$ as the exponent instead of n.

Example A Think-Pair-Share, Debriefing This example verifies that the formula results in the desired sum.

ELL Support

Students have encountered a lot of math terms that sound very similar: *arithmetic series* and *arithmetic sequence*, *arithmetic sequence* and *geometric sequence*, *common difference* and *common ratio*, etc. Provide linguistic support through translations of these terms and other language. As appropriate, remind students to refer to the English-Spanish glossary to aid their comprehension. Finally, monitor classroom discussions for understanding and correct language use.

My Notes

MATH TERMS

A **finite series** is the sum of a finite sequence and has a specific number of terms.

An ***infinite series*** is the sum of an infinite sequence and has an infinite number of terms. You will work with infinite series later in this Lesson.

MATH TIP

When writing out a *sequence*, separate the terms with commas. A *series* is written out as an expression and the terms are separated by addition symbols. If a series has negative terms, then the series may be written with subtraction symbols.

Learning Targets:

- Derive the formula for the sum of a finite geometric series.
- Calculate the partial sums of a geometric series.

SUGGESTED LEARNING STRATEGIES: Close Reading, Vocabulary Organizer, Think-Pair-Share, Create Representations

The sum of the terms of a geometric sequence is a ***geometric series***. The sum of a ***finite geometric series*** where $r \neq 1$ is given by these formulas:

$$S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1}$$

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

1. To derive the formula, Step 1 requires multiplying the equation of the sum by $-r$. Follow the remaining steps on the left to complete the derivation of the sum formula.

Step 1 $S_n = a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1}$

$-rS_n = -a_1r - a_1r^2 - a_1r^3 - \ldots - a_1r^{n-1} - a_1r^n$

Step 2 Combine terms on each side of the equation (most terms will cancel out).

$S_n - rS_n = a_1 - a_1r^n$

Step 3 Factor out S_n on the left side of the equation and factor out a_1 on the right.

$S_n(1 - r) = a_1(1 - r^n)$

Step 4 Solve for S_n.

$S_n = \frac{a_1(1-r^n)}{(1-r)} = a_1\left(\frac{1-r^n}{1-r}\right)$

Example A

Find the total of the Area of Square column in the table in Item 1 from the last lesson. Then use the formula developed in Item 1 of this lesson to find the total area and show that the result is the same.

Step 1: Add the areas of each square from the table.

$256 + 64 + 16 + 4 + 1 = 341$

Square #	1	2	3	4	5
Area	256	64	16	4	1

Step 2: Find the common ratio.

$\frac{64}{256} = 0.25, \frac{16}{64} = 0.25, \frac{4}{16} = 0.25; r = 0.25$

Step 3: Substitute $n = 5$, $a_1 = 256$, and $r = 0.25$ into the formula for S_n.

$S_n = a_1\left(\frac{1-r^n}{1-r}\right); S_5 = 256\left(\frac{1-0.25^5}{1-0.25}\right)$

Step 4: Evaluate S_5.

$S_5 = 256\left(\frac{1-0.25^5}{1-0.25}\right) = 341$

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Lesson 20-2
Geometric Series

My Notes

Try These A

Find the indicated sum of each geometric series. Show your work.

a. Find S_5 for the geometric series with $a_1 = 5$ and $r = 2$. **155**

b. $256 + 64 + 16 + 4 + \ldots$; S_6 **341.25**

c. $\sum_{n=1}^{10} 2 \cdot 3^{n-1}$ **59,048**

MATH TIP

Recall that *sigma notation* is a shorthand notation for a series. For example:

$$\sum_{n=1}^{3} 8 \cdot 2^{n-1}$$
$$= 8(2)^{(1-1)} + 8(2)^{(2-1)} + 8(2)^{(3-1)}$$
$$= 8 \cdot 1 + 8 \cdot 2 + 8 \cdot 4$$
$$= 8 + 16 + 32$$
$$= 56$$

Check Your Understanding

2. Reason quantitatively. How do you determine if the common ratio in a series is negative?

3. Find the sum of the series $2 + 8 + 32 + 128 + 512$ using sigma notation.

Recall that the sum of the first n terms of a series is a *partial sum*. For some geometric series, the partial sums $S_1, S_2, S_3, S_4, \ldots$ form a sequence with terms that approach a limiting value. The limiting value is called the ***sum of the infinite geometric series***.

To understand the concept of an infinite sum of a geometric series, follow these steps.

- Start with a square piece of paper, and let it represent one whole unit.
- Cut the paper in half, place one piece of the paper on your desk, and keep the other piece of paper in your hand. The paper on your desk represents the first partial sum of the series, $S_1 = \frac{1}{2}$.
- Cut the paper in your hand in half again, adding one of the pieces to the paper on your desk and keeping the other piece in your hand. The paper on your desk now represents the second partial sum.
- Repeat this process as many times as you are able.

MATH TIP

If the terms in the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ get close to some constant as n gets very large, the constant is the limiting value of the sequence. For example, in the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{n}, \ldots$, the terms get closer to a limiting value of 0 as n gets larger.

4. Use appropriate tools strategically. Each time you add a piece of paper to your desk, the paper represents the next term in the geometric series.

a. As you continue the process of placing half of the remaining paper on your desk, what happens to the amount of paper on your desktop? **The amount of paper on the desk gets closer to 1, the amount represented by the original square.**

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MINI-LESSON: Partial Sums of Geometric Series

If students need additional help with finding partial sums of geometric series, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand sigma notation and can find and write a sum using it. Ask students to describe how the sigma notation for a geometric series is similar to and different from the sigma notation for an arithmetic series. For example, both contain a general term, an index of summation, and upper and lower limits; the general term for a geometric series has an exponent containing a variable.

Answers

2. The terms will have signs alternating between positive and negative.

3. $\sum_{j=1}^{5} 2(4)^{j-1}$

Teacher to Teacher

The final part of this lesson deals with infinite sums. The concept of the existence of an infinite sum is difficult for many students. A geometric series can have a finite or infinite number of terms. The sum S_n of the first n terms of an infinite geometric series is called the nth partial sum of the series.

If the sequence of partial sums $S_1, S_2, S_3, \ldots S_n, \ldots$ approaches some specific number S, then the geometric series is said to have that number S as its sum.

The paper-cutting activity provides a concrete example of partial sums of an infinite series approaching a specific value.

Developing Math Language

Discuss with students that a *partial sum of a series* is a type of *finite series*. This will help students to see how some of the new vocabulary terms are related. Reinforce students' acquisition of all vocabulary through regular reference to words in the text as well as words you and students place on the classroom Word Wall.

4a Use Manipulatives Students will realize that the longer they continue the process of adding half the remaining paper to their desks, the closer the amount of paper on their desks gets to a complete square, which represents one whole.

4b–d Create Representations, Debriefing Students will also see the limiting value of 1 by looking at the problem numerically and graphically.

CONNECT TO AP

To illustrate convergence and divergence, have students graph the partial sums for each infinite geometric series below.

$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$S_n = \frac{1}{16} + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots$

The graphs show that the first series converges and the second diverges.

My Notes

CONNECT TO AP

An infinite series whose partial sums continually get closer to a specific number is said to *converge*, and that number is called the *sum of the infinite series.*

4. b. Fill in the blanks to complete the partial sums for the infinite geometric series represented by the pieces of paper on your desk.

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \underline{\frac{1}{4}} = \underline{\frac{3}{4}}$

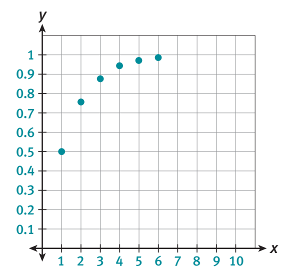
$S_3 = \frac{1}{2} + \underline{\frac{1}{4}} + \underline{\frac{1}{8}} = \underline{\frac{7}{8}}$

$S_4 = \frac{1}{2} + \underline{\frac{1}{4}} + \underline{\frac{1}{8}} + \underline{\frac{1}{16}} = \underline{\frac{15}{16}}$

$S_5 = \frac{1}{2} + \underline{\frac{1}{4}} + \underline{\frac{1}{8}} + \underline{\frac{1}{16}} + \underline{\frac{1}{32}} = \underline{\frac{31}{32}}$

$S_6 = \frac{1}{2} + \underline{\frac{1}{4}} + \underline{\frac{1}{8}} + \underline{\frac{1}{16}} + \underline{\frac{1}{32}} + \underline{\frac{1}{64}} = \underline{\frac{63}{64}}$

c. Plot the first six partial sums.



d. Do the partial sums appear to be approaching a limiting value? If so, what does the value appear to be?
Yes; the limiting value appears to be one.

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Lesson 20-2
Geometric Series

My Notes

5. Consider the geometric series $2 + 4 + 8 + 16 + 32 + \ldots$.
   a. List the first five partial sums for this series.
   $S_1 = 2$; $S_2 = 6$; $S_3 = 14$; $S_4 = 30$; $S_5 = 62$
   b. Do these partial sums appear to have a limiting value?
   No; there does not appear to be a limiting value.
   c. Does there appear to be a sum of the infinite series? If so, what does the sum appear to be? If not, why not?
   No; there does not appear to be an infinite sum. Since the terms that are being added in the partial sums are growing larger and larger, there will not be a limiting value on the sums.

6. Consider the geometric series $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \ldots$
   a. List the first seven partial sums for this series.
   $S_1 = 3$; $S_2 = 2$; $S_3 = \frac{7}{3} \approx 2.333$; $S_4 = \frac{20}{9} \approx 2.222$; $S_5 = \frac{61}{27} \approx 2.259$; $S_6 = \frac{182}{81} \approx 2.247$; $S_7 = \frac{547}{243} \approx 2.251$
   b. Do these partial sums appear to have a limiting value?
   Yes; about 2.25.
   c. Does there appear to be a sum of the infinite series? If so, what does the sum appear to be? If not, why not?
   Yes; the sum appears to be 2.25.

WRITING MATH

You can write the sum of an infinite series by using summation or sigma, notation and using an infinity symbol for the upper limit. For example,

$$\sum_{n=1}^{\infty} 3\left(-\frac{1}{3}\right)^{n-1} = 3 - 1 + \frac{1}{3} - \ldots$$

5–6 Look for a Pattern, Quickwrite, Debriefing These items provide one example in which the infinite sum of a geometric series does not exist (Item 5) and one in which the sum does exist (Item 6).

Teacher to Teacher

The study of infinite geometric series lays a foundation for the study of infinite series in calculus. In a calculus course, students will use the *n*th-term test for divergence, which states that if a sequence $\{a_n\}$ does not converge to 0, then the series $\sum a_n$ diverges. In other words, in Item 5, since the terms 2, 4, 8, 16, 32, … do not approach 0, the infinite sum 2 + 4 + 8 + 16 + 32 + … diverges, or does not exist.

In general, the converse of this test is *not* true, and thus it can never be used to prove convergence. In Item 6, you could not use the test to say that because the terms $3, -1, \frac{1}{3}, -\frac{1}{9}, \frac{1}{27}, \ldots$ approach 0, the sum $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27}, \ldots$ must exist.

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Check Your Understanding

Debrief students' answers to these items to ensure that they can write partial sums of an infinite series. Ask students to explain the relationship between the terms of a sequence and the partial sums.

Answers

7. 1; 3; 7; 15; 31; 63; 127; 255; no limit; no infinite sum
8. $\frac{2}{5}; \frac{8}{15}; \frac{26}{45}; \frac{80}{135}; \frac{242}{405}; \frac{728}{1215}$ which approximately equals 0.4; 0.533; 0.578; 0.593; 0.598; 0.599. There appears to be a limit and infinite sum of about 0.6.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 20-2 PRACTICE

9. $S_7 = 547$
10. $S_9 = 781.2496$
11. The sum is 0 when n is even.
12. The sum is -1 when n is odd.
13. The sums oscillate between 0 and -1.
14. When r is greater than or equal to 1, or less than or equal to -1, the partial sums do not appear to have a limiting value.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to find a particular partial sum without finding all the previous sums. If students make errors using the formula $S_n = a_1\left(\frac{1-r^n}{1-r}\right)$, be sure they are using the proper order of operations. For example, they must find the value of r^n before subtracting in the numerator. If they are using a calculator, they must be sure that the calculator subtracts in the numerator and in the denominator before dividing or multiplying; this will require either using parentheses or performing the calculation in several steps.

My Notes

Check Your Understanding

Find the indicated partial sums of each geometric series. Do these partial sums appear to have a limiting value? If so, what does the infinite sum appear to be?

7. First 8 partial sums of the series $1 + 2 + 4 + 8 + \ldots$
8. First 6 partial sums of the series $\frac{2}{5} + \frac{2}{15} + \frac{2}{45} + \frac{2}{135} + \ldots$

LESSON 20-2 PRACTICE

Find the indicated partial sum of each geometric series.

9. $1 - 3 + 9 - 27 + \ldots; S_7$
10. $\frac{1}{625} - \frac{1}{125} + \frac{1}{25} - \frac{1}{5} + \ldots; S_9$

Consider the geometric series $-1 + 1 - 1 + 1 - 1 + \ldots$

11. Find S_4 and S_6. Generalize the partial sum when n is an even number.
12. Find S_5 and S_7. Generalize the partial sum when n is on odd number.
13. Describe any conclusions drawn from Items 11 and 12.
14. **Construct viable arguments.** What conclusions if any can you draw from this lesson about the partial sums of geometric series where $r \geq 1$ or $r \leq -1$?

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Lesson 20-3
Convergence of Series

Learning Targets:
- Determine if an infinite geometric sum converges.
- Find the sum of a convergent geometric series.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite

Recall the formula for the sum of a finite series $S_n = \frac{a_1(1-r^n)}{1-r}$. To find the sum of an infinite series, find the value that S_n gets close to as n gets very large. For any infinite geometric series where $-1 < r < 1$, as n gets very large, r^n gets close to 0.

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

$$S \approx \frac{a_1(1-0)}{1-r}$$

$$\approx \frac{a_1}{1-r}$$

An infinite geometric series $\sum_{n=0}^{\infty} a_1 r^n$ converges to the sum $S = \frac{a_1}{1-r}$ if and only if $|r| < 1$ or $-1 < r < 1$. If $|r| \geq 1$, the infinite sum does not exist.

MATH TIP

S_n represents the sum of a finite series. Use S to indicate the sum of an infinite series.

MATH TIP

$-1 < r < 1$ can be written as $|r| < 1$. As n increases, r^n gets close to, or *approaches*, 0. It is important to realize that as r^n approaches 0, you can say that $|r^n|$, but not r^n, is getting "smaller."

1. Consider the three series from Items 4–6 of the previous lesson. Decide whether the formula for the sum of an infinite geometric series applies. If so, use it to find the sum. Compare the results to your previous answers.

a. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$ $S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$; the results are the same.

b. $2 + 4 + 8 + 16 + 32 + \ldots$ Does not apply since $r = 2 \geq 1$.

c. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - \ldots$

$S = \frac{3}{1-\left(-\frac{1}{3}\right)} = \frac{3}{\frac{4}{3}} = \frac{9}{4} = 2.25$; the results are the same.

My Notes

Check Your Understanding

Find the infinite sum if it exists or tell why it does not exist. Show your work.

2. $64 + 16 + 4 + 1 + \ldots$

3. $\frac{1}{3} + \frac{5}{12} + \frac{25}{48} + \frac{125}{192} + \ldots$

4. $\sum_{n=1}^{\infty} 3\left(\frac{2}{5}\right)^{n-1}$

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Answers

2. $S = \frac{256}{3} = 85.\overline{3}$

3. does not exist because $r = \frac{5}{4} \geq 1$

4. $S = 3 + \frac{6}{5} + \frac{12}{25} + \ldots = \frac{3}{\frac{3}{5}} = 5$

ACTIVITY 20 Continued

Lesson 20-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1
Check Your Understanding
#5–6
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find the indicated sum for each geometric series.

1. S_6: $3 + 0.3 + 0.03 + \ldots$ [3.33333]

2. S_5: $9 - 18 + 36 - 48 + \ldots$ [75]

3. S_6: $1 + 10 + 100 + 1000 + \ldots$ [11,111]

1 Create Representations, Debriefing Students use the formula for the sum of an infinite geometric series, $S = \frac{a_1}{1-r}$, $-1 < r < 1$, and validate the formula by comparing the results to series for which they have already examined the limit of the partial sums. Be sure students understand the restriction on r and that a sum does not exist if $|r| \geq 1$.

Some students may understand the formula better if they look at a numerical example first. When finding an infinite sum, you are finding the value that S_n gets close to as n gets very large. Consider the series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$. The nth partial sum is $S_n = \frac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}}$. As n gets very large, $\left(\frac{1}{3}\right)^n$ gets close to 0, and so

$$S_n \approx \frac{\frac{1}{3}(1-0)}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}.$$

Check Your Understanding

Debrief students' answers to these items to ensure that they know how to find infinite sums and how to determine whether an infinite sum exists. Watch for students who incorrectly determine whether a sum exists because they find the reciprocal of r instead of the correct value of r.

5–6 Look for a Pattern, Quickwrite, Think-Pair-Share, Debriefing Item 5 is included in case students question whether or not infinite arithmetic series have infinite sums.

Item 6 gives students the opportunity to reflect on all the formulas derived in the activity. After sharing answers with the entire group to make sure all responses are correct, students can record the information in their math notebooks.

Teacher to Teacher

All arithmetic series other than the trivial arithmetic series $0 + 0 + 0 + \ldots$ diverge by the *n*th-term test.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand and can verify that the series is geometric. These items make a connection between infinite geometric series and repeating decimals, which students have prior experience with. After Item 8, ask students to find the decimal value of $\frac{2}{9}$ and compare it to the series.

Answers

7. $r = 0.10$
8. yes; $S = \frac{a_1}{1-r} = \frac{0.2}{1-0.1} = \frac{0.2}{0.9} = \frac{2}{9}$
9. $\frac{5}{9}$, $S = \frac{a_1}{1-r} = \frac{0.5}{1-0.1} = \frac{5}{9}$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 20-3 PRACTICE

10. $S = 12$
11. $S = 2187$
12. does not exist because $r = \frac{4}{3} \geq 1$
13. does not exist; arithmetic sequence with terms growing larger in absolute value
14. Sample answer: If r is -0.5, r^2 is 0.25, which is closer to zero than r. As r is raised to greater and greater powers, the result gets closer to zero.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to determine whether an infinite sum exists for a geometric series. Watch for students who attempt to find an infinite sum without first checking whether it exists. Remind them that infinite sums may not exist, and demonstrate using examples from this activity.

---

5. Consider the arithmetic series $2 + 5 + 8 + 11 + \ldots$.
   a. Find the first four partial sums of the series.
   $S_1 = 2$; $S_2 = 7$; $S_3 = 15$; $S_4 = 26$
   b. Do these partial sums appear to have a limiting value?
   No; there does not appear to be a limiting value.
   c. Does the arithmetic series appear to have an infinite sum? Explain.
   No; there does not appear to be an infinite sum. Since the terms that are being added in the partial sums are growing larger and larger, there will not be a limiting value on the sums.

6. Summarize the following formulas for a geometric series.

common ratio	$r = \frac{a_n}{a_{n-1}}$
*n*th term	$a_n = a_1 \cdot r^{n-1}$
Sum of first *n* terms	$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$
Infinite sum	$S = \frac{a_1}{1-r}$ if $-1 < r < 1$

Check Your Understanding

Consider the series $0.2 + 0.02 + 0.002 + \ldots$.

7. Find the common ratio between the terms of the series.
8. Does this series have an infinite sum? If yes, use the formula to find the sum.
9. **Construct viable arguments.** Make a conjecture about the infinite sum $0.5 + 0.05 + 0.005 + \ldots$. Then verify your conjecture with the formula.

LESSON 20-3 PRACTICE

Find the infinite sum if it exists, or tell why it does not exist.

10. $18 - 9 + \frac{9}{2} - \frac{9}{4} + \ldots$
11. $729 + 486 + 324 + 216 + \ldots$
12. $81 + 108 + 144 + 192 + \ldots$
13. $-33 - 66 - 99 - 132 - \ldots$
14. **Reason quantitatively.** At the beginning of the lesson it is stated that "for any infinite geometric series where $-1 < r < 1$, as n gets very large, r^n gets close to 0." Justify this statement with an example, using a negative value for r.

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ACTIVITY 20 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 20-1

1. Write *arithmetic, geometric,* or *neither* for each sequence. If arithmetic, state the common difference. If geometric, state the common ratio.
   a. 4, 12, 36, 108, 324, . . .
   b. 1, 2, 6, 24, 120, . . .
   c. 4, 9, 14, 19, 24, . . .
   d. 35, −30, 25, −20, 15, . . .
2. Find the indicated term of each geometric series.
   a. $a_1 = 1, r = -3; a_{10}$
   b. $a_1 = 3072, r = \frac{1}{4}; a_8$
3. If a_n is a geometric sequence, express the quotient of $\frac{a_7}{a_4}$ in terms of r.
4. The first three terms of a geometric series are $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$. What is a_6?
   A. $\frac{3}{81}$
   B. 3
   C. $\frac{364}{81}$
   D. 9
5. Determine the first three terms of a geometric sequence with a common ratio of 2 and defined as follows:
   $x - 1, x + 6, 3x + 4$
6. Determine whether each sequence is geometric. If it is a geometric sequence, state the common ratio.
   a. $x, x^2, x^4, \ldots$
   b. $(x + 3), (x + 3)^2, (x + 3)^3, \ldots$
   c. $3^x, 3^{x+1}, 3^{x+2}, \ldots$
   d. $x^2, (2x)^2, (3x)^2, \ldots$
7. If $a_3 = \frac{9}{32}$ and $a_5 = \frac{81}{512}$, find a_1 and r.
8. The 5 in the expression $a_n = 4(5)^{n-1}$ represents which part of the expression?
   A. n
   B. a_1
   C. r
   D. S_n
9. A ball is dropped from a height of 24 feet. The ball bounces to 85% of its previous height with each bounce. Write an expression and solve to find how high (to the nearest tenth of a foot) the ball bounces on the sixth bounce.
10. Write the recursive formula for each sequence.
    a. 4, 2, 1, 0.5, . . .
    b. 2, 6, 18, 54, 120, . . .
    c. $\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \ldots$
    d. $-45, 5, -\frac{5}{9}, \ldots$
11. Write the explicit formula for each sequence.
    a. 4, 2, 1, 0.5, . . .
    b. 2, 6, 18, 54, 120, . . .
    c. $\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \ldots$
    d. $-45, 5, -\frac{5}{9}, \ldots$

Lesson 20-2

12. Find the indicated partial sum of each geometric series.
    a. $5 + 2 + \frac{4}{5} + \frac{8}{25} + \ldots; S_7$
    b. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \ldots; S_{15}$
13. For the geometric series 2.9 + 3.77 + 4.90 + 6.37 + . . . , do the following:
    a. Find S_9 (to the nearest hundredth).
    b. How many more terms have to be added in order for the sum to be greater than 200?
14. George and Martha had two children by 1776, and each child had two children. If this pattern continued to the 12th generation, how many descendants do George and Martha have?
15. A finite geometric series is defined as 0.6 + 0.84 + 1.12 + 1.65 + . . . + 17.36. How many terms are in the series?
    A. $n = 5$
    B. $n = 8$
    C. $n = 10$
    D. $n = 11$
16. Evaluate $\sum_{j=1}^{6} 3(2)^j$

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ACTIVITY 20 Continued

ACTIVITY PRACTICE

1. **a.** geometric; $r = 3$
   b. neither
   c. arithmetic; $d = 5$
   d. neither
2. **a.** $a_{10} = -19{,}683$
   b. $a_8 = \frac{3}{16}$, or 0.1875
3. r^3
4. B
5. $x = 8$; first three terms are 7, 14, 28
6. **a.** not geometric
   b. yes; $r = (x + 3)$
   c. yes; $r = 3$
   d. not geometric
7. $a_1 = \frac{1}{2}$ and $r = \frac{3}{4}$
8. C
9. $a_6 = 20.4(0.85)^{6-1} = 9.1$ ft
10. **a.** $a_n = \frac{1}{2}a_{n-1}$
    b. $a_n = 3a_{n-1}$
    c. $a_n = \frac{1}{5}a_{n-1}$
    d. $a_n = -\frac{1}{9}a_{n-1}$
11. **a.** $a_n = 4\left(\frac{1}{2}\right)^{n-1}$
    b. $a_n = 2(3)^{n-1}$
    c. $a_n = \frac{4}{5}\left(\frac{1}{5}\right)^{n-1}$
    d. $a_n = 45\left(-\frac{1}{9}\right)^{n-1}$
12. **a.** $S_7 = \frac{25{,}999}{3125} = 8.31968$
    b. $S_{15} = 4095.875$
13. **a.** $S_9 = 92.84$
    b. 3 more terms; $S_{12} = 215.55$
14. George and Martha are the first generation; they have 4,096 descendants at the 12th generation.
15. D
16. 378

17. a. \$80; \$5120
b. \$10,230
18. a. $a_n = 3 \cdot 3^{n-1} = 3^n$
b. 81
c. 121
19. a. $S = 48$
b. does not exist since $r = 2 \geq 1$
c. $\frac{7776}{7} \approx 1110.857$
20. $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$
21. $\frac{3}{11}$
22. a. $r = \frac{2}{3}$; series converges
b. $r = -\frac{1}{2}$; series converges
c. $r = 1.5$; series diverges
23. B
24. $r = \frac{1}{4}$
25. B
26. True. Sample answer: The terms in an arithmetic series are added to form partial sums. Since there is a common difference not equal to 0, the partial sum changes at a constant rate as terms are added, so there is no limiting value on the sums.
27. Sample answer: The common ratio can be found between any two terms by determining the number of terms between the given terms and rewriting the ratio as a quantity to that power. Working backward from the first term will give you a value for a_1.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

17. During a 10-week summer promotion, a baseball team is letting all spectators enter their names in a weekly drawing each time they purchase a game ticket. Once a name is in the drawing, it remains in the drawing unless it is chosen as a winner. Since the number of names in the drawing increases each week, so does the prize money. The first week of the contest the prize amount is \$10, and it doubles each week.
a. What is the prize amount in the fourth week of the contest? In the tenth week?
b. What is the total amount of money given away during the entire promotion?

18. In case of a school closing due to inclement weather, the high school staff has a calling system to make certain that everyone is notified. In the first round of phone calls, the principal calls three staff members. In the second round of calls, each of those three staff members calls three more staff members. The process continues until all of the staff is notified.
a. Write a rule that shows how many staff members are called during the nth round of calls.
b. Find the number of staff members called during the fourth round of calls.
c. If all of the staff has been notified after the fourth round of calls, how many people are on staff at the high school, including the principal?

Lesson 20-3

19. Find the infinite sum if it exists. If it does not exist, tell why.
a. $24 + 12 + 6 + 3 + \ldots$
b. $\frac{1}{12} + \frac{1}{6} + \frac{1}{3} + \frac{2}{3} + \ldots$
c. $1296 - 216 + 36 - 6 + \ldots$

20. Write an expression in terms of a_n that means the same as $\sum_{j=1}^{\infty} 2\left(\frac{1}{3}\right)^j$

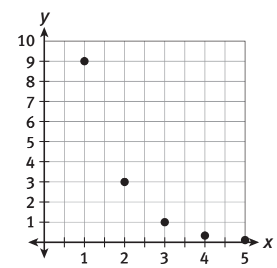
21. Express 0.2727 . . . as a fraction.

22. Use the common ratio to determine if the infinite series converges or diverges.
a. $36 + 24 + 12 + \ldots$
b. $-4 + 2 + (-1) + \ldots$
c. $3 + 4.5 + 6.75 + \ldots$

23. The infinite sum $0.1 + 0.05 + 0.025 + 0.0125 + \ldots$
A. diverges.
B. converges at 0.2.
C. converges at 0.5.
D. converges at 1.0.

24. An infinite geometric series has $a_1 = 3$ and a sum of 4. Find r.

25. The graph depicts which of the following?



A. converging arithmetic series
B. converging geometric series
C. diverging arithmetic series
D. diverging geometric series

26. True or false? No arithmetic series with a common difference that is not equal to zero has an infinite sum. Explain.

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

27. Explain how knowing any two terms of a geometric sequence is sufficient for finding the other terms.

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Sequences and Series
THE CHESSBOARD PROBLEM

Embedded Assessment 1
Use after Activity 20

In a classic math problem, a king wants to reward a knight who has rescued him from an attack. The king gives the knight a chessboard and plans to place money on each square. He gives the knight two options. Option 1 is to place a thousand dollars on the first square, two thousand on the second square, three thousand on the third square, and so on. Option 2 is to place one penny on the first square, two pennies on the second, four on the third, and so on.

Think about which offer sounds better and then answer these questions.

1. List the first five terms in the sequences formed by the given options. Identify each sequence as arithmetic, geometric, or neither.
   a. Option 1
   b. Option 2
2. For each option, write a rule that tells how much money is placed on the nth square of the chessboard and a rule that tells the total amount of money placed on squares 1 through n.
   a. Option 1
   b. Option 2
3. Find the amount of money placed on the 20th square of the chessboard and the total amount placed on squares 1 through 20 for each option.
   a. Option 1
   b. Option 2
4. There are 64 squares on a chessboard. Find the total amount of money placed on the chessboard for each option.
   a. Option 1
   b. Option 2
5. Which gives the better reward, Option 1 or Option 2? Explain why.

Embedded Assessment 1

Assessment Focus

- Identifying terms in arithmetic and geometric sequences
- Identifying common differences and common ratios
- Writing implicit and explicit rules for arithmetic and geometric sequences

Answer Key

1. **a.** 1000, 2000, 3000, 4000, 5000; arithmetic
   b. 0.01, 0.02, 0.04, 0.08, 0.16; geometric
2. **a.** $a_n = 1000 + (n - 1)1000$; $S_n = \frac{n}{2}(1000 + a_n)$
   b. $a_n = 0.01(2)^{n-1}$; $S_n = 0.01\left(\frac{1-2^n}{1-2}\right)$ or $0.01(2^n - 1)$
3. **a.** \$20,000; \$210,000
   b. \$5242.88; \$10,485.75
4. **a.** \$2,080,000
   b. about $\$1.845 \times 10^{17}$
5. Option 2 is better. The first term in the arithmetic series is much greater than the first term in the geometric series. However, the geometric series grows faster than the arithmetic series. At some point between the 20th and 64th terms, the corresponding terms in the geometric series will be greater than those in the arithmetic series.

Common Core State Standards for Embedded Assessment 1

HSA-SSE.B.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems.★
HSF-BF.A.1	Write a function that describes a relationship between two quantities.★
HSF-BF.A.1a	Determine an explicit expression, a recursive process, or steps for calculation from a context.
HSF-BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

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Teacher to Teacher

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Unpacking Embedded Assessment 2

Once students have completed this Embedded Assessment, turn to Embedded Assessment 2 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 2.

Embedded Assessment 1
Use after Activity 20

Sequences and Series
THE CHESSBOARD PROBLEM

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 3, 4)	• Fluency in determining specified terms of a sequence or the sum of a specific number of terms of a series	• A functional understanding and accurate identification of specified terms of a sequence or the sum of a specific number of terms of a series	• Partial understanding and partially accurate identification of specified terms of a sequence or the sum of a specific number of terms of a series	• Little or no understanding and inaccurate identification of specified terms of a sequence or the sum of a specific number of terms of a series
Problem Solving (Items 3, 4)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2)	• Fluency in accurately representing real-world scenarios with arithmetic and geometric sequences and series	• Little difficulty in accurately representing real-world scenarios with arithmetic and geometric sequences and series	• Some difficulty in representing real-world scenarios with arithmetic and geometric sequences and series	• Significant difficulty in representing real-world scenarios with arithmetic and geometric sequences and series
Reasoning and Communication (Item 5)	• Clear and accurate explanation of which option provides the better reward	• Adequate explanation of which option provides the better reward	• Misleading or confusing explanation of which option provides the better reward	• Incomplete or inadequate explanation of which option provides the better reward

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Exponential Functions and Graphs

ACTIVITY 21

Sizing Up the Situation

Lesson 21-1 Exploring Exponential Patterns

My Notes

Learning Targets:

- Identify data that grow exponentially.
- Compare the rates of change of linear and exponential data.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Quickwrite

Ramon Hall, a graphic artist, needs to make several different-sized draft copies of an original design. His original graphic design sketch is contained within a rectangle with a width of 4 cm and a length of 6 cm. Using the office copy machine, he magnifies the original 4 cm × 6 cm design to 120% of the original design size, and calls this his first draft. Ramon's second draft results from magnifying the first draft to 120% of its new size. Each new draft is 120% of the previous draft.

MATH TIP

Magnifying a design creates similar figures. The ratio between corresponding lengths of similar figures is called the *constant of proportionality*, or the *scale factor*. For a magnification of 120%, the scale factor is 1.2.

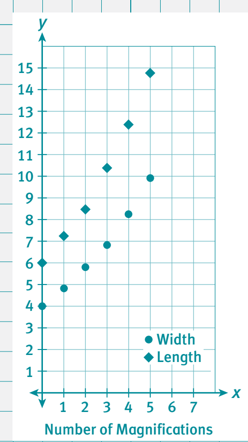
1. Complete the table with the dimensions of Ramon's first five draft versions, showing all decimal places.

Number of Magnifications	Width (cm)	Length (cm)
0	4	6
1	4.8	7.2
2	5.76	8.64
3	6.912	10.368
4	8.2944	12.4416
5	9.95328	14.92992

2. Make sense of problems. The resulting draft for each magnification has a unique width and a unique length. Thus, there is a functional relationship between the number of magnifications *n* and the resulting width *W*. There is also a functional relationship between the number of magnifications *n* and the resulting length *L*. What are the reasonable domain and range for these functions? Explain.

The reasonable domain for both functions is the nonnegative integers, because *n* represents a number of magnifications and thus cannot be negative or a decimal or fraction. The reasonable range for both functions is the nonnegative real numbers, because *W* and *L* represent measurements and so cannot be negative.

3. Plot the ordered pairs (n, W) from the table in Item 1. Use a different color or symbol to plot the ordered pairs (n, L).



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Common Core State Standards for Activity 21

HSF-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.
HSF-IF.B.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.
HSF-IF.C.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
HSF-IF.C.7e	Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

ACTIVITY 21

Investigative

Activity Standards Focus

In Activity 21, students examine exponential functions and their graphs. They begin by investigating linear growth and decay and compare rates of change in exponential and linear data. Next, they learn to write exponential functions. They perform transformations of the parent exponential function and, finally, they examine base exponential functions.

Students will rely on prior knowledge to investigate rates of change of exponential functions and to perform transformations of the parent function. Review transformations of previous types of functions, including quadratic and cubic.

Lesson 21-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–5 #6–8

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find the slope of the line represented by each equation.

1. $y = -3x + 5$ $[-3]$

2. $8x + 3y = 24$ $\left[-\frac{8}{3}\right]$

3. $y - 7 = 5(2x + 3)$ $[10]$

1–5 Create Representations, Look for a Pattern, Quickwrite, Debriefing In Item 1, check to see that students follow instructions and do not round in their computations.

As students complete Item 3, they may attempt to plot the points in a straight line. Encourage them to draw their points as accurately as possible.

Also, be alert to students who try to connect the points on their graphs with lines. Remind them of their answers to Item 2, in which they explained why a reasonable domain and range for both functions is discrete.

Differentiating Instruction

Have students who are having difficulty determining whether the points form a line use a ruler to attempt drawing lines through each set of points in Item 3.

Teacher to Teacher

This lesson calls on students' prior knowledge of linear functions and their constant rate of change. Students will discover that exponential functions, on the other hand, do not have a constant rate of change between any two points.

1–5 (continued) Create Representations, Look for a Pattern, Quickwrite, Debriefing Make sure students understand that the table they complete for Item 4 allows them to determine the rate of change for two functions. In both cases, the domain is the number of magnifications, and the change in the domain value represented in each row is 1.

6–8 Look for a Pattern, Quickwrite, Debriefing The intent of Item 6 is to review linear functions. In Part a, the data presented are clearly linear, since both Δx and Δy are constant. Confirming that the data in Part b are linear is more challenging for the students. They must verify that $\frac{\Delta y}{\Delta x}$ is constant for all pairs of data presented in the table.

The intent of Item 7 is for students to work with data that are neither linear nor quadratic. Students should notice that the pattern in this table is a simple multiplicative relationship in which the next entry in the table for y is one-half of the previous entry. After debriefing this item, have any students who thought the functions were linear go back and reconsider their answers.

My Notes

4. Use the data in Item 1 to complete the table.

Increase in Number of Magnifications	Change in the Width	Change in the Length
0 to 1	$4.8 - 4 = 0.8$	$7.2 - 6 = 1.2$
1 to 2	$5.76 - 4.8 = 0.96$	$8.64 - 7.2 = 1.44$
2 to 3	$6.912 - 5.76 = 1.152$	$10.368 - 8.64 = 1.728$
3 to 4	$8.2944 - 6.912 = 1.3824$	$12.4416 - 10.368 = 2.0736$
4 to 5	$9.95328 - 8.2944 = 1.65888$	$14.92992 - 12.4416 = 2.48832$

MATH TIP

Linear functions have the property that the rate of change of the output variable y with respect to the input variable x is constant, that is, the ratio $\frac{\Delta y}{\Delta x}$ is constant for linear functions.

5. From the graphs in Item 3 and the data in Item 4, do these functions appear to be linear? Explain why or why not.
The functions do not appear to be linear because there is not a constant rate of change and the graphs are not lines.

6. Express regularity in repeated reasoning. Explain why each table below contains data that can be represented by a linear function. Write an equation to show the linear relationship between x and y.

a.

x	−3	−1	1	3	5
y	8	5	2	−1	−4

Sample answer: The data are linear because there is a constant rate of change; y changes by −3 units for every 2 units of change in x. The linear equation is $y = -1.5x + 3.5$.

$y =$ ____________________

b.

x	2	5	11	17	26
y	3	7	15	23	35

Sample answer: The data are linear because the ratios $\frac{\Delta y}{\Delta x}$ are constant, $\frac{\Delta y}{\Delta x} = \frac{4}{3}$, for all pairs of data in the table. The linear equation is $y = \frac{4}{3}x + \frac{1}{3}$.

$y =$ ____________________

7. Consider the data in the table below.

x	0	1	2	3	4
y	24	12	6	3	1.5

a. Can the data in the table be represented by a linear function? Explain why or why not.
No, the data cannot be represented by a linear function because the y-values change by a different amount for each unit change in x.

b. Describe any patterns that you see in the consecutive y-values.
Each y-value in the table is $\frac{1}{2}$ of the previous entry.

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Common Core State Standards for Activity 21 *(continued)*

HSF-BF.A.1 Write a function that describes a relationship between two quantities.★

HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.

HSF-BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Lesson 21-1
Exploring Exponential Patterns

ACTIVITY 21
continued

My Notes

8. Consider the data in the table in Item 1. How does the relationship of the data in this table compare to the relationship of the data in the table in Item 7?
Sample answer: Both patterns are formed by multiplying each term by a constant to get the next term. In Item 7 the constant multiplier is $\frac{1}{2}$, and in Item 1 the constant multiplier is 1.2.

Check Your Understanding

9. Complete the table so that the function represented is a linear function.

x	1	2	3	4	5
$f(x)$	16	22			40

10. Reason quantitatively. Explain why the function represented in the table cannot be a linear function.

x	1	2	3	4	5
$f(x)$	7	12	16	19	21

LESSON 21-1 PRACTICE

Model with mathematics. Determine whether each function is linear or nonlinear. Explain your answers.

11. x = number of equally sized pans of brownies; $f(x)$ = number of brownies

12. x = cost of an item; $f(x)$ = price you pay in a state with a 6% sales tax

13. x = number of months; $f(x)$ = amount of money in a bank account with interest compounded monthly

14.

x	2	4	6	8	10
y	2.6	3.0	3.8	4.8	6.0

15.

x	5	10	15	20	25
y	1.25	1.00	0.75	0.50	0.25

16. Identify if there is a constant rate of change or constant multiplier. Determine the rate of change or constant multiplier.

x	1	2	3	4
y	6	4.8	3.84	3.072

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6–8 (continued) Look for a Pattern, Quickwrite, Debriefing Item 8 returns to the numerical pattern of Item 1. Students should realize that they cannot calculate consecutive y-values by adding a constant amount each time the x-value increases by 1.

Check Your Understanding
Debrief students' answers to these items to ensure that they understand the difference between a linear function and an exponential function and can use the information in a table to identify each type of function.

Answers

9.

x	1	2	3	4	5
$f(x)$	16	22	28	34	40

10. Sample answer: The difference between the first two $f(x)$ values, 12 and 7, is 5. If the function is linear, then 5 would be the common difference and the following ordered pairs would be (3, 17), (4, 22), and (5, 27), which is not the case.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 21-1 PRACTICE

11. linear
12. linear
13. nonlinear
14. nonlinear
15. linear
16. constant multiplier; 0.8

ADAPT

Check students' answers to the Lesson Practice to ensure that they can differentiate between linear and nonlinear functions given a description of a domain and its corresponding range. Encourage students to write the functions for Items 11–13. Once they have done so, they can create a table of values or graph the function to help them decide whether it is linear.

Lesson 21-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–4
#5–6 #7
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find the value of each expression when $x = 2$.

1. 4^x [16]
2. x^{-4} $\left[\frac{1}{16}\right]$
3. $(x + 2)^x$ [16]
4. -3^x [−9]

1–2 Close Reading, Vocabulary Organizer, Quickwrite, Look for a Pattern, Create Representations, Debriefing Students focus on the concept of an exponential function and learn its distinguishing features. When the change in x is constant, as it is in the table in Item 2, the y-values of an exponential function change by a constant multiplicative amount.

Completing the table in Item 2 should assist students in recognizing an exponential pattern, and so they can write the exponential function $W(n) = 4(1.2)^n$.

Teacher to Teacher

When students use the Connect to Technology, the issue of domain may come up. If it does not come up here, revisit it in Item 2 of Lesson 21-3. The domain of the problem situation is a subset of the nonnegative integers. The functions graphed on the calculator will have a domain of all real numbers.

Technology Tip

Students without a graphing calculator may evaluate the expressions in the Calculation to Find Width column for 0 to 5 magnifications to confirm that the function is reasonable. For additional technology resources, visit SpringBoard Digital.

My Notes

MATH TERMS

An **exponential function** is a function of the form $f(x) = a \cdot b^x$, where a and b are constants, x is the domain, $f(x)$ is the range, and $a \neq 0$, $b > 0$, $b \neq 1$.

MATH TERMS

In an exponential function, the constant multiplier, or scale factor, is called an **exponential decay factor** when the constant is less than 1.

When the constant is greater than 1, it is called an **exponential growth factor**.

MATH TIP

To compare change in size, you could also use the *growth rate*, or *percent increase*. This is the percent that is equal to the ratio of the increase amount to the original amount.

CONNECT TO TECHNOLOGY

Confirm the reasonableness of your function in Item 2b by using a graphing calculator to make a scatter plot of the data in the table in Item 8 in Lesson 21-1. Then graph the function to see how it compares to the scatter plot.

Learning Targets:

- Identify and write exponential functions.
- Determine the decay factor or growth factor of an exponential function.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Create Representations, Look for a Pattern, Quickwrite, Think-Pair-Share

The data in the tables in Items 7 and 8 of the previous lesson were generated by ***exponential functions***. In the special case when the change in the input variable x is constant, the output variable y of an exponential function changes by a *multiplicative constant*. For example, in the table in Item 7, the increase in the consecutive x-values results from repeatedly adding 1, while the decrease in y-values results from repeatedly multiplying by the constant $\frac{1}{2}$, known as the ***exponential decay factor***.

1. In the table in Item 1 in Lesson 21-1, what is the ***exponential growth factor***?
   1.2

2. You can write an equation for the exponential function relating W and n.
   a. Complete the table below to show the calculations used to find the width of each magnification.

Number of Magnifications	Calculation to Find Width (cm)
0	4
1	4(1.2)
2	4(1.2)(1.2)
3	4(1.2)(1.2)(1.2)
4	4(1.2)(1.2)(1.2)(1.2)
5	4(1.2)(1.2)(1.2)(1.2)(1.2)
10	$4(1.2)^{10}$
n	$4(1.2)^n$

   b. **Express regularity in repeated reasoning.** Write a function that expresses the resulting width W after n magnifications of 120%.
   $W(n) = 4(1.2)^n$

   c. Use the function in part b to find the width of the 11th magnification.
   $W(11) \approx 29.720$

Developing Math Language

Review ***exponential expressions*** and compare and contrast them to ***exponential functions.*** Discuss the difference between growth and decay to help students understand the meanings of ***exponential growth factors*** and ***exponential decay factors.***

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Lesson 21-2
Exponential Functions

My Notes

The general form of an exponential function is $f(x) = a(b^x)$, where a and b are constants and $a \neq 0$, $b > 0$, $b \neq 1$.

3. For the exponential function written in Item 2b, identify the value of the parameters a and b. Then explain their meaning in terms of the problem situation.
a represents the initial width of 4 cm, and b represents the growth factor of 1.2.

4. Starting with Ramon's original 4 cm × 6 cm rectangle containing his graphic design, write an exponential function that expresses the resulting length L after n magnifications of 120%.
$L(n) = 6(1.2)^n$

Ramon decides to print five different reduced draft copies of his original design rectangle. Each one will be reduced to 90% of the previous size.

5. Complete the table below to show the dimensions of the first five draft versions. Include all decimal places.

Number of Reductions	Width (cm)	Length (cm)
0	4	6
1	3.6	5.4
2	3.24	4.86
3	2.916	4.374
4	2.6244	3.9366
5	2.36196	3.54294

6. Write the exponential decay factor and the *decay rate* for the data in the table in Item 5.
The exponential decay factor is 0.9 and the decay rate is 10%.

MATH TIP

To compare change in size, you could also use the *decay rate*, or *percent decrease*. This is the percent that is equal to the ratio of the decrease amount to the original amount.

7. Model with mathematics. Use the data in the table in Item 5.

a. Write an exponential function that expresses the width w of a reduction in terms of n, the number of reductions performed.
$w(n) = 4(0.9)^n$

b. Write an exponential function that expresses the length l of a reduction in terms of n, the number of reductions performed.
$l(n) = 6(0.9)^n$

c. Use the functions to find the dimensions of the design if the original design undergoes ten reductions.
$w(n) \approx 1.395$; $l(n) \approx 2.092$

3–4 Quickwrite, Create Representations, Debriefing In Item 3, students will explain the meaning of parameters of the equation in terms of the problem situation.

Some students may extend their response in Item 3 to a new resizing situation and simply write the function using $a = 6$ cm and $b = 1.2$. Other students may replicate the work in Item 2 by making a table of values.

Differentiating Instruction

To **support** students' writing efforts, you may want to add a section to your Word Wall on basic sentence structure in English writing (simple sentence, compound sentence, complex sentence, transition words, etc.). Review these structures with students prior to the writing assignment, and provide an opportunity to clarify any questions about language structures.

5–6 Create Representations, Vocabulary Organizer, Think-Pair-Share In Item 5, check to see that students follow instructions and do not round numbers in their computations.

The intent of Item 6 is to distinguish between decay factor and decay rate. The Math Tip can help guide the students.

TEACHER TO TEACHER

In this section of the activity, students investigate exponential decay in the context of the original problem. Instead of repeated magnifications of 120%, this second considers repeated reductions of 90%.

7 Create Representations, Think-Pair-Share, Debriefing Students may extend their response in Item 3 to a new resizing situation and simply write the functions using the appropriate values for a and b.

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Check Your Understanding

Debrief students' answers to these items to ensure that they can write an exponential function and identify the meaning of *a* and *b* in context.

Answers

8. Sample answer: If $a = 0$, the function would be $f(x) = 0$, a constant function. If $b = 1$, the function would be $f(x) = 1$, also a constant function. If $b < 0$, the function would not be continuous.
9. a. 2
   b. Exponential growth; The function values are increasing.
10. $a = 2000$; $b = 1.05$; $2000 is deposited in an account with an annual interest rate of 5%.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 21-2 PRACTICE

11. Exponential; *x* increases by a constant amount while *y* increases at a rate of 3^x; $y = 3^x$.
12. Neither; *x* increases by a constant amount; *y* increases, but not by a constant amount or a constant multiplier.
13.

x	0	1	2	3	4
y	64	51.2	40.96	32.768	26.2144

14. 20%
15. $f(x) = 64(0.8)^x$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand the difference between exponential growth and exponential decay. Discuss how they can use a table to determine whether the function represents growth or decay.

Check Your Understanding

8. Why is it necessary to place restrictions that $a \neq 0$, $b > 0$, and $b \neq 1$ in the general form of an exponential function?
9. An exponential function contains the ordered pairs (3, 6), (4, 12), and (5, 24).
   a. What is the scale factor for this function?
   b. Does the function represent exponential decay or growth? Explain your reasoning.
10. **Make sense of problems.** For the equation $y = 2000(1.05)^x$, identify the value of the parameters *a* and *b*. Then explain their meaning in terms of a savings account in a bank.

LESSON 21-2 PRACTICE

Construct viable arguments. Decide whether each table of data can be modeled by a linear function, an exponential function, or neither, and justify your answers. If the data can be modeled by a linear or exponential function, give an equation for the function.

11.

x	0	1	2	3	4
y	1	3	9	27	81

12.

x	0	1	2	3	4
y	4	8	14	22	32

13. Given that the function has an exponential decay factor of 0.8, complete the table.

x	0	1	2	3	4
y	64	51.2	40.96	32.768	26.2144

14. What is the decay rate for the function in Item 13?
15. Write the function represented in Item 13.

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Lesson 21-3
Exponential Graphs and Asymptotes

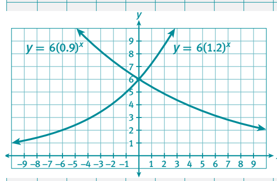
ACTIVITY 21 *continued*

My Notes

Learning Targets:
- Determine when an exponential function is increasing or decreasing.
- Describe the end behavior of exponential functions.
- Identify asymptotes of exponential functions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Activating Prior Knowledge, Close Reading, Vocabulary Organizer, Think-Pair-Share, Group Presentation

1. Graph the functions $y = 6(1.2)^x$ and $y = 6(0.9)^x$ on a graphing calculator or other graphing utility. Sketch the results.



2. Determine the domain and range for each function. Use interval notation.

	Domain	Range
a. $y = 6(1.2)^x$	$(-\infty, \infty)$	$(0, \infty)$
b. $y = 6(0.9)^x$	$(-\infty, \infty)$	$(0, \infty)$

A function is said to *increase* if the *y*-values increase as the *x*-values increase. A function is said to *decrease* if the *y*-values decrease as the *x*-values increase.

3. Describe each function as increasing or decreasing.
   a. $y = 6(1.2)^x$
   increasing
   b. $y = 6(0.9)^x$
   decreasing

CONNECT TO AP

Not all functions increase or decrease over the entire domain of the function. Functions may increase, decrease, or remain constant over various intervals of the domain. Functions that either increase or decrease over the entire domain are called *strictly monotonic*.

As you learned in a previous activity, the *end behavior* of a graph describes the *y*-values of the function as *x* increases without bound and as *x* decreases without bound. If the end behavior approaches some constant *a*, then the graph of the function has a horizontal ***asymptote*** at $y = a$.

When *x* increases without bound, the values of *x* approach positive infinity, ∞. When *x* decreases without bound, the values of *x* approach negative infinity, $-\infty$.

MATH TERMS

If the graph of a relation gets closer and closer to a line, the line is called an **asymptote** of the graph.

4. Describe the end behavior of each function as *x* approaches ∞. Write the equation for any horizontal asymptotes.
   a. $y = 6(1.2)^x$
   As x goes to infinity, y goes to infinity.
   b. $y = 6(0.9)^x$
   As x goes to infinity, y gets close to 0; there is a horizontal asymptote at $y = 0$.

1–2 Create Representations, Activating Prior Knowledge, Debriefing Students should use a graphing calculator or other graphing utility for Item 1.

Students should note that the domain of the functions in the problem situation representing enlargements and reductions are subsets of the counting numbers, whereas the domain of the function in Items 2a and 2b are all real numbers. Students can explore the range of each function by using their graphing calculators.

3 Activating Prior Knowledge, Debriefing Students should have an intuitive knowledge of when a function is increasing and when it is decreasing. Ask students to describe what the graph of an increasing function looks like and what the graph of a decreasing function looks like.

Lesson 21-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3 #4–7
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

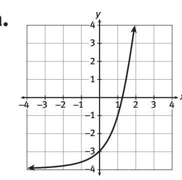
Ask students to find the domain and range of each function.

1. $f(x) = \frac{2}{x}$ [*domain: all real numbers except 0; range: all real numbers except 0*]
2. $f(x) = x^2 + 1$ [*domain: all real numbers; range: all real numbers greater than or equal to 1*]
3. $f(x) = x^2 - 2x + 4$ [*domain: all real numbers; range: all real numbers greater than or equal to 3*]

Differentiating Instruction

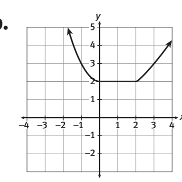
For advanced learners who wish to further explore the Connect to AP, ask students to identify the intervals on which each function shown below is increasing, decreasing, and/or constant. Students should also identify any functions that are strictly monotonic.

a.



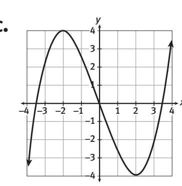
[*increasing for all reals; strictly monotonic*]

b.



[*decreasing for $x < 0$; constant for $0 < x < 2$; increasing for $x > 2$*]

c.



[*decreasing for $-2 < x < 2$; increasing for $x < -2$ and $x > 2$*]

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4–7 Create Representations, Think-Pair-Share, Group Presentation, Debriefing Use Items 4–7 to assess student understanding of the concepts of end behavior, ∞, $-\infty$, and asymptotes as related to the graphs of exponential functions.

Students should be able to answer Item 6 by referring to the graphs of the functions.

Item 7 is intended to spark a class discussion of the features of the graphs of exponential functions as related to the parameters of the function. Some students may not think about the possibility of the value of a being negative. Try to find a group to consider this and have them present their findings to the class.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the meaning of *increasing* and *decreasing* and can identify the domain, range, and asymptotes of an exponential function.

Answers

8. domain (both): $(-\infty, \infty)$; range (both): $(-\infty, 0)$
9. $f(x)$: y approaches $-\infty$; $g(x)$: y approaches 0
10. $f(x)$: y approaches 0; $g(x)$: y approaches $-\infty$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 21-3 PRACTICE

11. Increases because $a > 0$ and $b > 1$; y-intercept is (0, 8) because $a = 8$ gives the value of the y-intercept.
12. Decreases because $a > 0$ and $0 < b < 1$; y-intercept is (0, 0.3) because $a = 0.3$ gives the value of the y-intercept.
13. Decreases because $a < 0$ and $b > 1$; y-intercept is $(0, -2)$ because $a = -2$ gives the value of the y-intercept.
14. Increases because $a < 0$ and $0 < b < 1$; y-intercept is $(0, -1)$ because $a = -1$ gives the value of the y-intercept.
15. The asymptotes are all $y = 0$; the y-intercepts are all $x = a$. These conditions are true for all exponential functions of the form $f(x) = a(b^x)$: when x becomes either very large or very small, $a(b^x)$ will approach 0; when $x = 0$, $f(x) = a$.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand when an exponential function is increasing and when it is decreasing by looking at the function definition. Students can graph each function to confirm their results.

ACTIVITY 21 *continued*

My Notes

Lesson 21-3
Exponential Graphs and Asymptotes

5. Describe the end behavior of each function as x approaches $-\infty$. Write the equation for any horizontal asymptotes.
   a. $y = 6(1.2)^x$
   As x goes to negative infinity, y gets close to 0; there is a horizontal asymptote at $y = 0$.
   b. $y = 6(0.9)^x$
   As x goes to negative infinity, y goes to infinity.

6. Identify any x- or y-intercepts of each function.
   a. $y = 6(1.2)^x$
   x-intercept: none; y-intercept: (0, 6)
   b. $y = 6(0.9)^x$
   x-intercept: none; y-intercept: (0, 6)

7. **Reason abstractly.** Consider how the parameters a and b affect the graph of the general exponential function $f(x) = a(b)^x$. Use a graphing calculator to graph f for various values of a and b.
   a. When does the function increase?
   for $a > 0$ and $b > 1$ or for $a < 0$ and $0 < b < 1$
   b. When does the function decrease?
   for $a > 0$ and $0 < b < 1$ or for $a < 0$ and $b > 1$
   c. What determines the y-intercept of the function?
   the value of a
   d. State any horizontal asymptotes of the function.
   $y = 0$

Check Your Understanding

Graph the functions $f(x) = -6(1.2)^x$ and $g(x) = -6(0.9)^x$ on a graphing calculator or other graphing utility.

8. Determine the domain and range for each function.
9. Describe the end behavior of each function as x approaches ∞.
10. Describe the end behavior of each function as x approaches $-\infty$.

LESSON 21-3 PRACTICE

Make use of structure. For each exponential function, state whether the function increases or decreases, and give the y-intercept. Use the general form of an exponential function to explain your answers.

11. $y = 8(4)^x$
12. $y = 0.3(0.25)^x$
13. $y = -2(3.16)^x$
14. $y = -(0.3)^x$
15. **Construct viable arguments.** What is true about the asymptotes and y-intercepts of the functions in this lesson? What conclusions can you draw?

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Learning Targets:

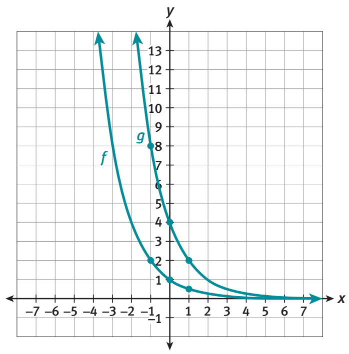
- Explore how changing parameters affects the graph of an exponential function.
- Graph transformations of exponential functions.

SUGGESTED LEARNING STRATEGIES: Close Reading, Create Representations, Quickwrite

You can use transformations of the graph of the function $f(x) = b^x$ to graph functions of the form $g(x) = a(b)^{x-c} + d$, where a and b are constants, and $a \neq 0$, $b > 0$, $b \neq 1$. Rather than having a single parent graph for all exponential functions, there is a different parent graph for each base b.

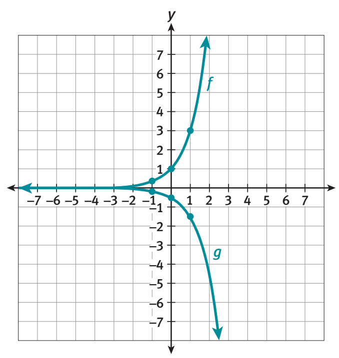
1. Graph the parent graph f and the function g by applying the correct vertical stretch, shrink, and/or reflection over the x-axis. Write a description of each transformation.

a. $f(x) = \left(\frac{1}{2}\right)^x$ $\qquad g(x) = 4\left(\frac{1}{2}\right)^x$



The graph of g is a vertical stretch of the graph of f by a factor of 4.

b. $f(x) = 3^x$ $\qquad g(x) = -\frac{1}{2}(3)^x$



The graph of g is a vertical shrink of the graph of f by a factor of $\frac{1}{2}$ and a reflection over the x-axis.

My Notes

MATH TIP

You can draw a quick sketch of the parent graph for any base b by plotting the points $\left(-1, \frac{1}{b}\right)$, (0, 1), and (1, b).

CONNECT TO AP

Exponential functions are important in the study of calculus.

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ACTIVITY 21 Continued

Lesson 21-4

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–4
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to describe each transformation of the graph of the parent function $f(x) = x^2$.

1. $g(x) = (x + 3)^2 - 1$ [*shifted 3 units left and 1 unit down*]
2. $g(x) = -2x^2$ [*reflection across x-axis and a vertical stretch by a factor of 2*]
3. $g(x) = 3(x - 1)^2$ [*vertical stretch by a factor of 3 and shifted 1 unit right*]

TEACHER to TEACHER

In this lesson, students will explore graphing exponential functions through transformations. Use the Bell-Ringer Activity as a review of transformations.

1–2 Create Representations, Activating Prior Knowledge, Debriefing Students should see that the constant a can cause a vertical stretch or shrink of the parent graph. When $a < 0$, there is a reflection over the x-axis.

CONNECT TO AP

Exponential functions play an important part in the study of calculus and have some very interesting properties. The instantaneous rate of change (derivative) of any exponential function in the form $f(x) = b^x$ is a multiple of the function f. This makes sense when one considers that the slope of an exponential function is always increasing at an increasing rate as x increases. A function that fits this description well is an exponential function. Even more surprising is that the derivative of the exponential function $f(x) = e^x$—a special exponential function that is discussed in the next lesson—is the function itself. This is the only nonzero function that has the property of being its own derivative.

1–2 (continued) Create Representations, Quickwrite, Debriefing Students should see that for $a(b)^{x-c}+d$, the constant c causes a horizontal translation of the parent graph, and the constant d causes a vertical translation of the parent graph.

Technology Tip

Discuss asymptotic behavior as it relates to graphs of exponential functions. Explain that in Item 2a, it looks like the graph touches the x-axis but, in fact, it never does. Have students graph the function using an online calculator or geometry software. Have them zoom in to the x-axis, where they will see that the function does not cross or intersect it.

My Notes

CONNECT TO TECHNOLOGY

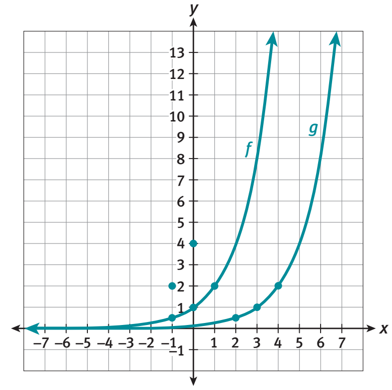
You can use a graphing calculator to approximate the range values when the x-coordinates are not integers. For $f(x) = 2^x$, use a calculator to find $f\left(\frac{1}{2}\right)$ and $f\left(\sqrt{3}\right)$.

$$2^{\frac{1}{2}} \approx 1.414$$
$$2^{\sqrt{3}} \approx 3.322$$

Then use a graphing calculator to verify that the points $\left(\frac{1}{2}, 2^{\frac{1}{2}}\right)$ and $\left(\sqrt{3}, 2^{\sqrt{3}}\right)$ lie on the graph of $f(x) = 2^x$.

2. Sketch the parent graph f and the graph of g by applying the correct horizontal or vertical translation. Write a description of each transformation and give the equations of any asymptotes.

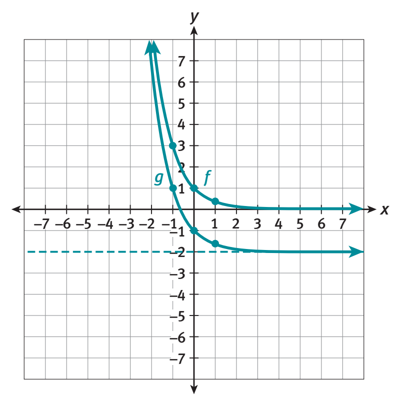
a. $f(x) = 2^x$
$g(x) = 2^{(x-3)}$



The graph of g is a horizontal translation of the graph of f to the right 3 units. Asymptote for f: $y = 0$; asymptote for g: $y = 0$

b. $f(x) = \left(\frac{1}{3}\right)^x$

$g(x) = \left(\frac{1}{3}\right)^x - 2$



The graph of g is a vertical translation of the graph of f down 2 units. Asymptote for f: $y = 0$; asymptote for g: $y = -2$

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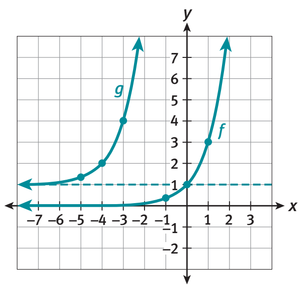
Lesson 21-4
Transforming Exponential Functions

My Notes

3. Attend to precision. Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. Give the domain and range of each function in interval notation. Give the equations of any asymptotes.

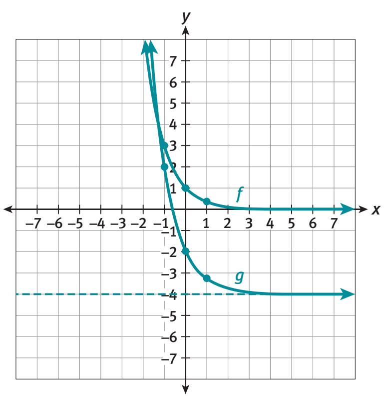
a. $g(x) = 3^{x+4} + 1$



To obtain the graph of *g*, horizontally translate the graph of *f* left 4 units and then vertically translate up 1 unit.

	f	*g*
Domain:	$(-\infty, \infty)$	$(-\infty, \infty)$
Range:	$(0, \infty)$	$(1, \infty)$
Asymptotes:	$y = 0$	$y = 1$

b. $g(x) = 2\left(\frac{1}{3}\right)^x - 4$



To obtain the graph of *g*, vertically stretch the graph of *f* by a factor of 2 and then vertically translate down 4 units.

	f	*g*
Domain:	$(-\infty, \infty)$	$(-\infty, \infty)$
Range:	$(0, \infty)$	$(4, \infty)$
Asymptotes:	$y = 0$	$y = -4$

3–4 Create Representations, Quickwrite, Debriefing Item 3 is an opportunity for students to practice transformations of parent graphs of exponential functions and to assess their understanding.

Have students identify the domain and range of each function using its equation. Then have them use the graph to confirm their findings.

Review different ways to write the domain and range of a function. They can be written in words, in set notation, or in interval notation. Present students with examples of each to review. For example, for the function $f(x) = 2^x$, the domain and range can be expressed as:

Domain: all real numbers
Range: all real numbers greater than 0
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$
$D = \{x \mid x \in \Re\}$
$R = \{y \mid y > 0\}$

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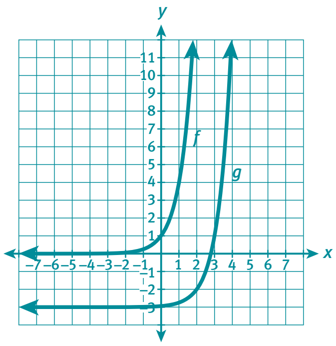
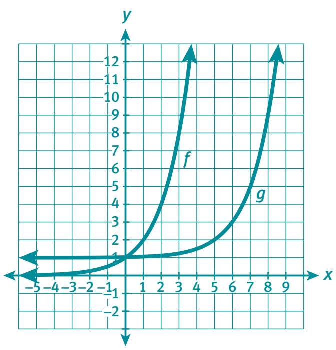
LESSON 21-4 PRACTICE

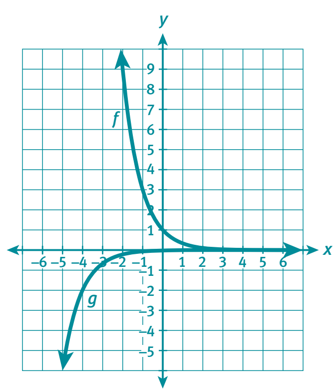
7. The parent graph is $f(x) = 4^x$; there is a horizontal translation 2 units to the right and a shift down 3 units; domain: $(-\infty, \infty)$; range: $(-3, \infty)$; asymptote: $y = -3$.



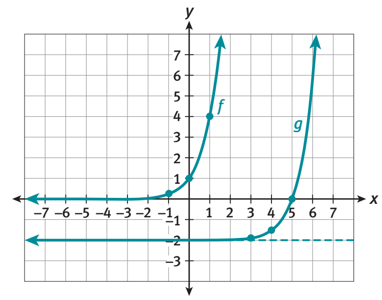
8. The parent graph is $f(x) = 2^x$; there is a vertical shrink of $\frac{1}{2}$, a horizontal translation 4 units to the right, and a vertical shift up 1 unit; domain: $(-\infty, \infty)$; range: $(1, \infty)$; asymptote: $y = 1$.



9. The parent graph is $f(x) = \left(\frac{1}{3}\right)^x$; there is a reflection over the x-axis, a vertical stretch by a factor of 2, and a horizontal translation left 4 units; vertical shift up 1 unit; domain: $(-\infty, \infty)$; range: $(-\infty, 0)$; asymptote: $y = 0$.



c. $g(x) = \frac{1}{2}(4)^{x-4} - 2$

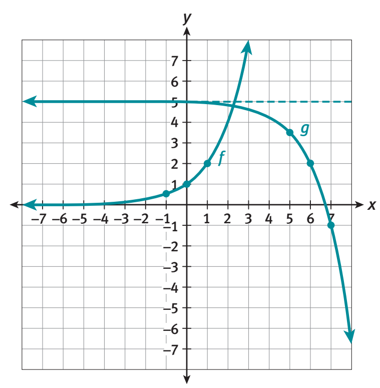


To obtain the graph of *g*, vertically shrink the graph of *f* by a factor of $\frac{1}{2}$, translate horizontally to the right 4 units, and then translate vertically down 2 units.

	f	*g*
Domain:	$(-\infty, \infty)$	$(-\infty, \infty)$
Range:	$(0, \infty)$	$(-2, \infty)$
Asymptotes:	$y = 0$	$y = -2$

4. Describe how the function $g(x) = -3(2)^{x-6} + 5$ results from transforming a parent graph $f(x) = 2^x$. Sketch both graphs on the same axes. Give the domain and range of each function in interval notation. Give the equations of any asymptotes. Use a graphing calculator to check your work.

To obtain the graph of *g*, reflect the graph of *f* over the *x*-axis, vertically stretch the graph of *f* by a factor of 3, and then vertically translate up 5 units and horizontally translate 6 units to the right.



	f	*g*
Domain:	$(-\infty, \infty)$	$(-\infty, \infty)$
Range:	$(-\infty, \infty)$	$(-\infty, 5)$
Asymptotes:	$y = 0$	$y = 5$

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Lesson 21-4
Transforming Exponential Functions

Check Your Understanding

5. **Reason quantitatively.** Explain how to change the equation of a parent graph $f(x) = 4^x$ to a translation that is left 6 units and a vertical shrink of 0.5.
6. Write the parent function $f(x)$ of $g(x) = -3(2)^{(x+2)} - 1$ and describe how the graph of $g(x)$ is a translation of the parent function.

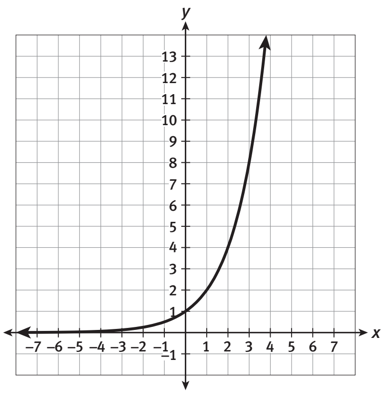
LESSON 21-4 PRACTICE

Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

7. $g(x) = 4^{x-2} - 3$
8. $g(x) = \frac{1}{2}(2)^{x-4} + 1$
9. $g(x) = -2\left(\frac{1}{3}\right)^{x+4}$

Make use of structure. Write the equation that indicates each transformation of the parent equation $f(x) = 2^x$. Then use the graph below and draw and label each transformation.

10. For $g(x)$, the y-intercept is at (0, 3).
11. For $h(x)$, the exponential growth factor is 0.5.
12. For $k(x)$, the graph of $f(x)$ is horizontally translated to the right 3 units.
13. For $l(x)$, the graph of $f(x)$ is vertically translated upward 2 units.



My Notes

ACTIVITY 21 Continued

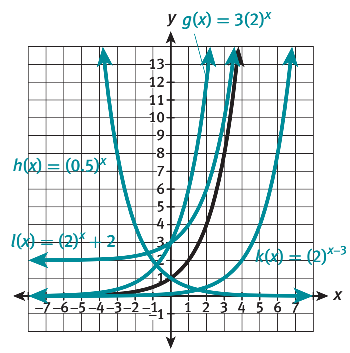
Check Your Understanding
Debrief students' answers to these items to ensure that they understand how to describe the transformation of an exponential quadratic function.

Answers
5. Replace a with 0.5, c with -6, and d with 0 in the equation $a(b)^{x-c} + d$.
6. The parent function is $f(x) = 2^x$; $g(x)$ is a translation of $f(x)$ in that it is reflected over the x-axis, vertically stretched by a factor of 3, horizontally translated left 2 units, and vertically translated up 1 unit.

ASSESS
Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 21-4 PRACTICE
7–9. See page 334.
10. $g(x) = 3(2)^x$
11. $h(x) = (0.5)^x$
12. $k(x) = (2)^{x-3}$
13. $l(x) = (2)^x + 2$



ADAPT
Check students' answers to the Lesson Practice to ensure that they can identify attributes of a transformation from an equation. Encourage students to graph the functions in Items 7–9 to check their answers.

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Lesson 21-5

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–3 #4–6

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to determine whether each expression evaluates to a number that is rational or irrational.

1. $\pi + 2$ [*irrational*]
2. $\sqrt{4} + \sqrt{3}$ [*irrational*]
3. $\frac{1}{3} + \frac{4}{3}$ [*rational*]

Technology Tip

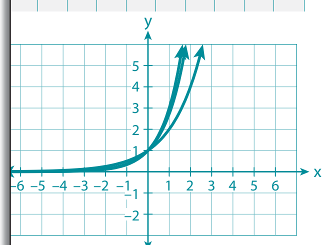
If students are unfamiliar with how to use a graphing calculator, walk them through Item 1. Students should consult their manuals if they are using a calculator other than a TI-Nspire. For some students, writing this process in their notes will be helpful, as they can refer to it repeatedly as they work through the course. For additional technology resources, visit SpringBoard Digital.

1–3 Create Representations, Quickwrite, Debriefing Students should understand that because e is an irrational number, e^x is an irrational number when x does not equal 0. In the table for Item 3, have students identify which values are rational and which are irrational.

My Notes

MATH TIP

Exponential functions that describe examples of (continuous) exponential growth or decay use *e* for the base. You will learn more about the importance of *e* in Precalculus.



Learning Targets:

- Graph the function $f(x) = e^x$.
- Graph transformations of $f(x) = e^x$.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Group Presentation, Debriefing

1. **Use appropriate tools strategically.** On a graphing calculator, set $Y_1 = x$ and $Y_2 = \left(1 + \frac{1}{x}\right)^x$. Let x increase by increments of 100. Describe what happens to the table of values for Y_2 as x increases.
   Sample answer: The values of Y_2 approach a constant value, ≈ 2.7183.

This irrational constant is called *e* and is often used in exponential functions.

2. a. On a graphing calculator, enter $Y_1 = e^x$. Using the table of values associated with Y_1, complete the table below.

x	$Y_1 = e^x$
0	1
1	2.7183
2	7.3891
3	20.086

   b. **Reason quantitatively.** Which row in the table gives the approximate value of *e*? Explain.
   The row in which $x = 1$; when $x = 1$, $Y_1 = e^1 = e$.

   c. What kind of number does *e* represent?
   an irrational number

3. a. Complete the table below.

x	x^{-1}	x^0	x^1	x^2	x^3
2	0.5	1	2	4	8
e	0.3679	1	2.7183	7.3891	20.086
3	0.3333	1	3	9	27

   b. Graph the functions $f(x) = e^x$, $g(x) = 2^x$, and $h(x) = 3^x$ on the same coordinate plane.

   c. Compare $f(x)$ with $g(x)$ and $h(x)$. Which features are the same? Which are different?
   All three functions have the *y*-intercept (0, 1) and the horizontal asymptote $y = 0$. The graph of the function $f(x)$ is between the graphs of $g(x)$ and $h(x)$, which makes sense because $2 < e < 3$.

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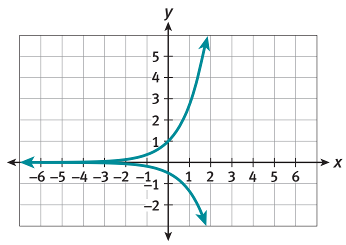
Lesson 21-5
Natural Base Exponential Functions

My Notes

4. Graph the parent function $f(x) = e^x$ and the function $g(x)$ by applying the correct vertical stretch, shrink, reflection over the x-axis, or translation. Write a description for the transformation. State the domain and range of each function. Give the equation of any asymptotes.

a. $f(x) = e^x$

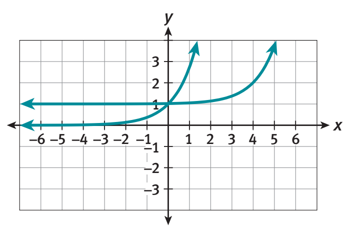
$g(x) = -\frac{1}{2}(e^x)$



The graph of g is a vertical shrink of the graph of f by a factor of $\frac{1}{2}$ and a reflection over the x-axis. The domain of both functions is $(-\infty, \infty)$. The range of f is $(0, \infty)$; the range of g is $(-\infty, 0)$. The asymptote of both functions is $y = 0$.

b. $f(x) = e^x$

$g(x) = e^{x-4} + 1$



The graph of g is a horizontal translation to the right 4 units and a vertical translation up 1 unit of f. The domain of both functions is $(-\infty, \infty)$. The range of f is $(0, \infty)$; the range of g is $(1, \infty)$. The asymptote of f is $y = 0$; the asymptote of g is $y = 1$.

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4 Create Representations, Quickwrite, Debriefing Refer back to transformations of exponential functions in the previous lesson, and have students predict what the transformations for Items 4a and 4b will look like before graphing. Discuss the similarities and differences between quadratic and cubic transformations and exponential transformations.

Developing Math Language

As students respond to questions or discuss possible solutions to problems, monitor their use of new terms and descriptions of applying math concepts to ensure their understanding and ability to use language correctly and precisely.

Universal Access

Have students use an online calculator or a geometry tool to graph the parent function and its transformation. With these tools, they can drag the functions to help them answer Items 4–6.

5 Create Representations, Quickwrite, Debriefing Debrief students' answers to Item 5b by having them identify each transformation and the reason for it. For example, the parent function $f(x)$ is translated down 2 units by the -2 in $g(x)$.

Discuss how each transformation in Item 5 affects the domain and range of the function. Ask students why the domain stays the same for each transformation and why the range changes.

ELL Support

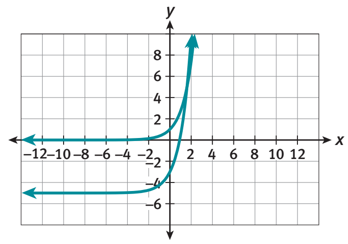
Students may struggle with the vocabulary in this lesson. Items 4–5 require students to describe transformations using math language. Have students refer to the Word Wall or their notes to help them articulate their answers.

My Notes

5. Graph the parent graph f and the function g by applying the correct transformation. Write a description of each transformation. State the domain and range of each function. Give the equation of any asymptotes.

a. $f(x) = e^x$

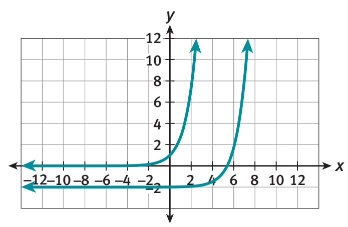
$g(x) = 2e^x - 5$



The graph of g is a vertical stretch of the graph of f by a factor of 2 and a vertical translation down 5 units. The domain of both functions is $(-\infty, \infty)$. The range of f is $(0, \infty)$; the range of g is $(-5, \infty)$. The asymptote of f is $y = 0$; the asymptote of g is $y = -5$.

b. $f(x) = e^x$

$g(x) = \frac{1}{2}(e^{x-4}) - 2$



The graph of g is a vertical shrink of the graph of f by a factor of $\frac{1}{2}$, a horizontal translation right 4 units, and a vertical translation down 2 units. The domain of both functions is $(-\infty, \infty)$. The range of f is $(0, \infty)$; the range of g is $(-2, \infty)$. The asymptote of f is $y = 0$; the asymptote of g is $y = -2$.

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Lesson 21-5
Natural Base Exponential Functions

ACTIVITY 21 *continued*

My Notes

6. Explain how the parameters a, c, and d transform the parent graph $f(x) = b^x$ to produce the graph of the function $g(x) = a(b)^{x-c} + d$.

$|a| > 1 \Rightarrow$ a vertical stretch by a factor of a

$0 < |a| < 1 \Rightarrow$ a vertical shrink by a factor of a

$a < 0 \Rightarrow$ a reflection over the x-axis

$c > 0 \Rightarrow$ a horizontal translation right c units

$c < 0 \Rightarrow$ a horizontal translation left c units

$d < 0 \Rightarrow$ a vertical shift down d units

$d > 0 \Rightarrow$ a vertical shift up d units

Check Your Understanding

Match each exponential expression with its graph.

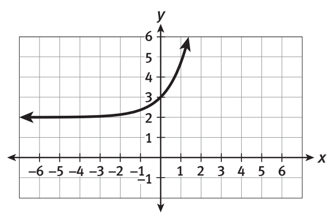
7. $f(x) = 3e^x$

8. $f(x) = -0.4e^x$

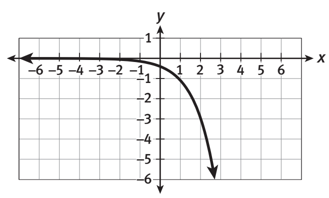
9. $f(x) = e^x + 2$

10. $f(x) = -e^x$

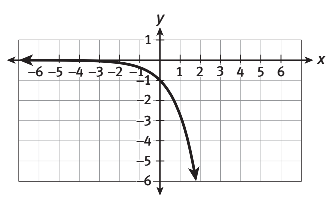
A.



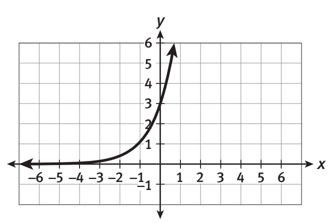
B.



C.



D.



ACTIVITY 21 Continued

6 Create Representations, Think-Pair-Share, Group Presentation, Debriefing Use Item 6 to assess student understanding of transformations of the graphs of all exponential functions. If students are having difficulty completing this item, suggest they substitute numbers for a, c, and d to help them. Have students work in pairs or groups to complete Item 6.

Monitor group discussions to ensure that all members of the group are participating and that each member understands the language and terms used in the discussion.

Check Your Understanding

Debrief students' answers to these items to ensure that they can identify key attributes of a natural base exponential function given an equation.

Answers

7. D
8. B
9. A
10. C

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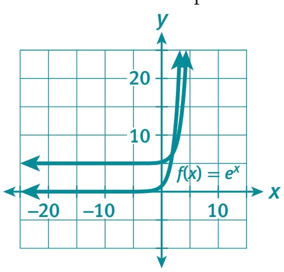
ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

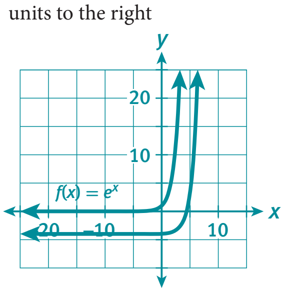
LESSON 21-5 PRACTICE

11. vertical shrink by a factor of $\frac{1}{4}$ and a translation 5 units up



	f	g
domain	$(-\infty, \infty)$	$(-\infty, \infty)$
range	$(0, \infty)$	$(5, \infty)$
asymptote	$y = 0$	$y = 5$

12. a translation 4 units down and 3 units to the right



	f	g
domain	$(-\infty, \infty)$	$(-\infty, \infty)$
range	$(0, \infty)$	$(-4, \infty)$
asymptote	$y = 0$	$y = -4$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to graph transformations of natural base exponential functions. Allow students to use graphing calculators or geometry to check their work.

ACTIVITY 21
continued

My Notes

Lesson 21-5
Natural Base Exponential Functions

LESSON 21-5 PRACTICE

Model with mathematics. Describe how each function results from transforming a parent graph of the form $f(x) = e^x$. Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.

11. $g(x) = \frac{1}{4}e^x + 5$

12. $g(x) = e^{x-3} - 4$

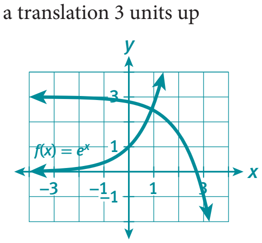
13. $g(x) = -4e^{x-3} + 3$

14. $g(x) = 2e^{x+4}$

15. **Critique the reasoning of others.** On Cameron's math test, he was asked to describe the transformations from the graph of $f(x) = e^x$ to the graph of $g(x) = e^{x-2} - 2$. Cameron wrote "translation left 2 units and down 2 units." Do you agree or disagree with Cameron? Explain your reasoning.

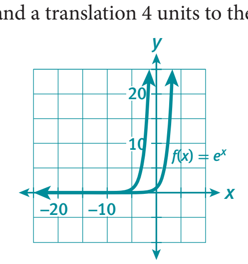
16. What similarities, if any, are there between the functions studied in this lesson and the previous lesson?

13. a reflection over the x-axis, a vertical stretch by a factor of 4, and a translation 3 units up



	f	g
domain	$(-\infty, \infty)$	$(-\infty, \infty)$
range	$(0, \infty)$	$(-\infty, 3)$
asymptote	$y = 0$	$y = 3$

14. a vertical stretch by a factor of 2 and a translation 4 units to the left



	f	g
domain	$(-\infty, \infty)$	$(-\infty, \infty)$
range	$(0, \infty)$	$(0, \infty)$
asymptote	$y = 0$	$y = 0$

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ACTIVITY 21 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 21-1

1. a. Complete the table so that the function represented is a linear function.

x	1	2	3	4	5
f(x)	5.4	6.7			10.6

b. What function is represented in the data?

2. a. How do you use a table of values to determine if the relationship of $y = 3x + 2$ is a linear relationship?
b. How do you use a graph to determine if the relationship in part a is linear?

3. Which relationship is nonlinear?
A. (2, 12), (5, 18), (6.5, 21)
B. $(6, x + 2), (21, x + 7), (-9, x - 3)$
C. (0.25, 1.25), (1.25, 2.50), (2.50, 5.00)
D. $(-5, 20), (-3, 12), (-1, 4)$

4. Determine if the table of data can be modeled by a linear function. If so, give an equation for the function. If not, explain why not.

x	0	1	2	3	4
y	$\frac{1}{5}$	$\frac{3}{5}$	1	$1\frac{2}{5}$	$1\frac{4}{5}$

5. Which relationship has the greatest value for $x = 4$?
A. $y = 5(3)^x + 2$
B. $y = 5(2^x + 3)$
C. $y = 5(3x + 2)$
D. $y = 5(2)^{x+3}$

6. Ida paints violets onto porcelain plates. She paints a spiral that is a sequence of violets, the size of each consecutive violet being a fraction of the size of the preceding violet. The table below shows the width of the first three violets in the continuing pattern.

Violet Number	1	2	3
Width (cm)	4	3.2	2.56

a. Is Ida's shrinking violet pattern an example of an exponential function? Explain.
b. Find the width of the fourth and fifth violets in the sequence.
c. Write an equation to express the size of the smallest violet in terms of the number of violets on the plate.
d. If a plate has a total of 10 violets, explain two different ways to determine the size of the smallest violet.

Lesson 21-2

7. Which statement is NOT true for the exponential function $f(x) = 4(0.75)^x$?
A. Exponential growth factor is 75%.
B. Percent of decrease is 25%.
C. The scale factor is 0.75.
D. The decay rate is 25%.

8. For the exponential function $f(x) = 3.2(1.5)^x$, identify the value of the parameters *a* and *b*. Then explain their meaning, using the vocabulary from the lesson.

9. Decide whether each table of data can be modeled by a linear function, an exponential function, or neither. If the data can be modeled by a linear or exponential function, give an equation for the function.

a.

x	0	1	2	3	4
y	24	18	12	6	0

b.

x	0	1	2	3	4
y	36	18	9	4.5	2.25

10. Sixteen teams play in a one-game elimination match. The winners of the first round go on to play a second round until only one team remains undefeated and is declared the champion.
a. Make a table of values for the number of rounds and the number of teams participating.
b. What is the reasonable domain and the range of this function? Explain.
c. Find the rate of decay.
d. Find the decay factor.

LESSON 21-5 PRACTICE (continued)

15. The correct answer is "translation right 2 units and down 2 units." Cameron likely made the error of thinking that subtracting 2 in the exponent results in a translation to the left because, on a number line, subtraction is usually associated with moving to the left.

16. Since $f(x) = e^x$ is an exponential function, the functions in both lessons are exponential functions and follow the same rules.

ACTIVITY 21 Continued

ACTIVITY PRACTICE

1. a.

x	1	2	3	4	5
f(x)	5.4	6.7	8.0	9.3	10.6

b. $f(x) = 1.3x + 4.1$

2. a. Sample answer: Subtract consecutive *y*-values to determine if there is a common difference.
b. Sample answer: The graph is a straight line.

3. C

4. yes; $y = \frac{2}{5}x + \frac{1}{5}$

5. D

6. a. Yes, it is an example of an exponential function because as the *x*-values increase by 1 each time, there is a constant ratio of 0.8 between the *y*-values.
b. $2.56(0.8) = 2.048$ cm; $2.048(0.8) = 1.6384$ cm
c. $y = 4(0.8)^{x-1}$ or $y = 5(0.8)^x$
d. You can substitute 10 into the function given in part c and have $y = 4\,(0.8)^{10-1} = 0.537$ cm rounded to three decimal places, or you can use a calculator and repeatedly multiply the starting violet width by 0.8 and find the same result.

7. A

8. $a = 3.2$, the initial value; $b = 1.5$, the growth factor or scale factor

9. a. linear; $y = -6x + 24$
b. exponential; $y = 36\left(\frac{1}{2}\right)^x$

10. a.

x	1	2	3	4	5
f(x)	16	8	4	2	1

b. domain: $\{x: 1, 2, 3, 4, 5\}$; range: $\{y: 16, 8, 4, 2, 1\}$
c. 50%
d. 0.50

11. A and C

12. B

13. a. $(-\infty, \infty)$; $(0, \infty)$; increases; (0, 2)
b. $(-\infty, \infty)$; $(0, \infty)$; decreases; (0, 3)
c. $(-\infty, \infty)$; $(-\infty, 0)$; increases; $(0, -1)$
d. $(-\infty, \infty)$; $(-\infty, 0)$; decreases; $(0, -3)$

14. a. 1.37%; 0.84%; 0.362%
b. $1.37\% - 0.84\% + 0.362\% = 0.892\%$
c. 100.892%
d. $P(n) = 313{,}847{,}465\,(1.00892)^n$, where P = population and n = years since 2012
e. $P(38) \approx 439{,}819{,}438$

15. $f(x)$ is increasing when $a > 0$.

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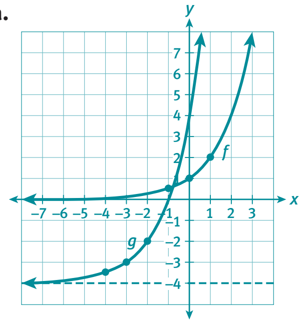
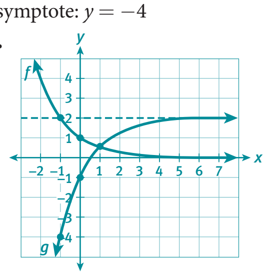
16. a.



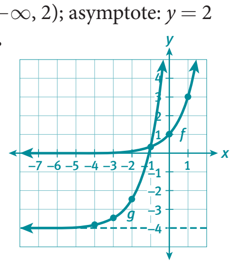
Translate the graph of f horizontally to the left 3 units and then vertically translate down 4 units.
f: domain $(-\infty, \infty)$; range $(0, \infty)$; asymptote: $y = 0$
g: domain $(-\infty, \infty)$; range $(-4, \infty)$; asymptote: $y = -4$

b.



Vertically stretch the graph of f by a factor of 3, reflect over the x-axis, and vertically translate up 2 units.
f: domain $(-\infty, \infty)$; range $(0, \infty)$; asymptotes: $y = 0$
g: domain $(-\infty, \infty)$; range $(-\infty, 2)$; asymptote: $y = 2$

c.



Vertically shrink the graph of f by a factor of $\frac{1}{2}$, horizontally translate left 3 units, and then vertically translate down 4 units.
f: domain $(-\infty, \infty)$; range $(0, \infty)$; asymptote: $y = 0$
g: domain $(-\infty, \infty)$; range $(-4, \infty)$; asymptote: $y = -4$

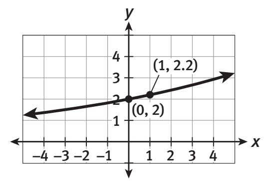
ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Lesson 21-3

11. Which of the following functions have the same graph?
A. $f(x) = \left(\frac{1}{4}\right)^x$
B. $f(x) = 4^x$
C. $f(x) = 4^{-x}$
D. $f(x) = x^4$

12. Which function is modeled in the graph below?



A. $y = (2)^x$
B. $y = 2(1.1)^x$
C. $y = (2)^{1.1x}$
D. $y = 2.1x$

13. For each exponential function, state the domain and range, whether the function increases or decreases, and the y-intercept.
a. $y = 2(4)^x$ **b.** $y = 3\left(\frac{1}{2}\right)^x$
c. $y = -(0.3)^x$ **d.** $y = -3(5.2)^x$

14. The *World Factbook* produced by the Central Intelligence Agency estimates the July 2012 United States population as 313,847,465. The following rates are also reported as estimates for 2012.

Birth rate: 13.7 births/1000 population
Death rate: 8.4 deaths/1000 population
Net migration rate: 3.62 migrant(s)/1000 population

a. Write a percent for each rate listed above.
b. Combine the percents from part a to find the overall growth rate for the United States.
c. The exponential growth factor for a population is equal to the growth rate plus 100%. What is the exponential growth rate for the United States?
d. Write a function to express the United States population as a function of years since 2012.
e. Use the function from part d to predict the United States population in the year 2050.

15. Under what conditions is the function $f(x) = a(3)^x$ increasing?

Lesson 21-4

16. Describe how each function results from transforming a parent graph of the form $f(x) = b^x$. Then sketch the parent graph and the given function on the same axes. State the domain and range of each function and give the equations of any asymptotes.
a. $g(x) = 2^{x+3} - 4$
b. $g(x) = -3\left(\frac{1}{2}\right)^x + 2$
c. $g(x) = \frac{1}{2}(3)^{x+3} - 4$

17. a. Explain why a change in c for the function $a(b)^{x-c} + d$ causes a horizontal translation.
b. Explain why a change in d for the function $a(b)^{x-c} + d$ causes a vertical translation.

18. Which transformation maps the graph of $f(x) = 3^x$ to $g(x) = \left(\frac{1}{3}\right)^x$?
A. horizontal translation
B. shrink
C. reflection
D. vertical translation

Lesson 21-5

19. Is $f(x) = e^x$ an increasing or a decreasing function? Explain your reasoning.

20. Which function has a y-intercept of (0, 0)?
A. $y = e^x + 1$
B. $y = -e^x + 1$
C. $y = e^x - 1$
D. $y = e^x$

21. What ordered pair do $f(x) = e^x$ and $g(x) = 2^x$ have in common?

MATHEMATICAL PRACTICES
Attend to Precision

22. Explain the difference between $y = x^2$ and $y = 2^x$.

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17. a. Sample answer: By adding to or subtracting from the x-value of the exponential term, the exponential term is being evaluated for a different value than x.
b. Sample answer: By adding to or subtracting from the exponential term containing x, the value of the function will be increased or decreased.

18. C
19. increasing; because $e > 1$
20. B
21. (0, 1)
22. Sample answer: The first is a polynomial function with a constant exponent and a changing base whose graph is a parabola. The second is an exponential function with a constant base and a changing exponent whose graph is nonlinear but is not a parabola.

Logarithms and Their Properties

ACTIVITY 22

Earthquakes and Richter Scale
Lesson 22-1 Exponential Data

My Notes

Learning Targets:
- Complete tables and plot points for exponential data.
- Write and graph an exponential function for a given context.
- Find the domain and range of an exponential function.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations, Quickwrite, Close Reading, Look for a Pattern

In 1935, Charles F. Richter developed the Richter magnitude test scale to compare the size of earthquakes. The Richter scale is based on the amplitude of the seismic waves recorded on seismographs at various locations after being adjusted for distance from the epicenter of the earthquake.

Richter assigned a magnitude of 0 to an earthquake whose amplitude on a seismograph is 1 micron, or 10^{-4} cm. According to the Richter scale, a magnitude 1.0 earthquake causes 10 times the ground motion of a magnitude 0 earthquake. A magnitude 2.0 earthquake causes 10 times the ground motion of a magnitude 1.0 earthquake. This pattern continues as the magnitude of the earthquake increases.

1. **Reason quantitatively.** How does the ground motion caused by earthquakes of these magnitudes compare?
   a. magnitude 5.0 earthquake compared to magnitude 4.0
      A magnitude 5.0 earthquake's ground motion is 10 times that of a magnitude 4.0 earthquake.
   b. magnitude 4.0 earthquake compared to magnitude 1.0
      A magnitude 4.0 earthquake's ground motion is 1000 or 10^3 times that of a magnitude 1.0 earthquake.
   c. magnitude 4.0 earthquake compared to magnitude 0
      A magnitude 4.0 earthquake's ground motion is 10,000 or 10^4 times that of a magnitude 0 earthquake.

The table below describes the effects of earthquakes of different magnitudes.

Typical Effects of Earthquakes of Various Magnitudes

1.0	Very weak, no visible damage
2.0	Not felt by humans
3.0	Often felt, usually no damage
4.0	Windows rattle, indoor items shake
5.0	Damage to poorly constructed structures, slight damage to well-designed buildings
6.0	Destructive in populated areas
7.0	Serious damage over large geographic areas
8.0	Serious damage across areas of hundreds of miles
9.0	Serious damage across areas of hundreds of miles
10.0	Extremely rare, never recorded

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Common Core State Standards for Activity 22

HSF-IF.B.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★
HSF-IF.C.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
HSF-IF.C.7e	Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

ACTIVITY 22
Investigative

Activity Standards Focus
In Activity 22, students examine logarithmic functions and their graphs. They begin reviewing exponential functions. Then they examine the relationship between logarithmic and exponential functions and write equations using both forms. Students discover and use the properties of logarithms and graph logarithmic functions.

Lesson 22-1

PLAN
Pacing: 1 class period

Chunking the Lesson

#1 #2 #3

Check Your Understanding

#6–7

Check Your Understanding

Lesson Practice

TEACH
Bell-Ringer Activity

Ask students to simplify each expression using properties of exponents.

1. $x^2y^{-3} \bullet xy^2$ $\left[\frac{x^3}{y}\right]$
2. $\left(2^3\right)^2$ [64]
3. $\frac{ab^5}{a^4b^{-1}}$ $\left[\frac{b^6}{a^3}\right]$
4. $\frac{3^5}{\left(3^2\right)^{-2}}$ [19,683]

Introduction, 1 Summarizing, Paraphrasing, Debriefing The table of typical effects will help students understand the physical implications of different magnitudes. Lead a class discussion before starting this lesson to make certain that all students are comfortable with earthquake terminology.

Use the table on the next page to assess student understanding of magnitude and ground motion caused by an earthquake.

Universal Access

Have students look up all the meanings of the word *magnitude* to help them understand its meaning with respect to earthquakes. Discuss if and where they have seen or heard the word used before.

2 Create Representations, Debriefing The purpose of this item is to lead students to the idea that there is an exponential relationship between ground motion caused by an earthquake and its magnitude.

3 Create Representations, Quickwrite, Debriefing Students experience how cumbersome it is to work with specific amounts of ground motion since the values vary so greatly. They should understand why Richter devised a scale using more manageable numbers. In part d, students should write an exponential function.

As students work on this item, monitor their writing to ensure that they are using language correctly, including adequate details, and describing mathematical reasoning using precise terms.

Technology Tip

Allow students to use a graphing calculator, an online calculator, or a geometry tool to create the graph in Item 3d. With any of these tools, students can easily change the displays to determine an appropriate window for displaying the function. For additional technology resources, visit SpringBoard Digital.

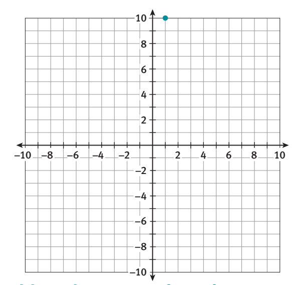
My Notes

2. Complete the table to show how many times as great the ground motion is when caused by each earthquake as compared to a magnitude 0 earthquake.

Magnitude	Ground Motion Compared to Magnitude 0
1.0	10
2.0	100
3.0	**1000**
4.0	**10,000**
5.0	**100,000**
6.0	**1,000,000**
7.0	**10,000,000**
8.0	**100,000,000**
9.0	**1,000,000,000**
10.0	**10,000,000,000**

3. In parts a–c below, you will graph the data from Item 2. Let the horizontal axis represent the magnitude of the earthquake and the vertical axis represent the amount of ground motion caused by the earthquake as compared to a magnitude 0 earthquake.

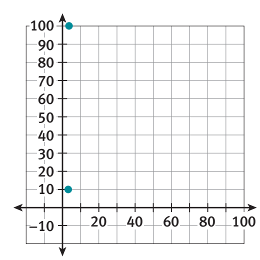
a. Plot the data using a grid that displays $-10 \le x \le 10$ and $-10 \le y \le 10$. Explain why this grid is or is not a good choice.



A [−10, 10] × [−10, 10] window would not be appropriate since only the ordered pair (1, 10) would be plotted on the graph, as shown.

b. Plot the data using a grid that displays $-10 \le x \le 100$ and $-10 \le y \le 100$. Explain why this grid is or is not a good choice.

A [−10, 100] × [−10, 100] window would not be appropriate since only the ordered pairs (1, 10) and (2, 100) would be plotted, as shown. In addition, the *x*-axis does not need to be larger than 10 units.



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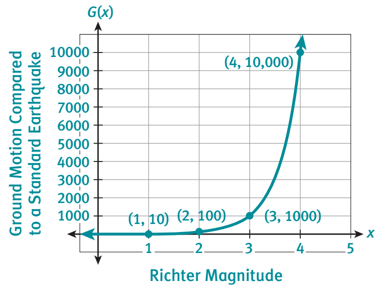
Lesson 22-1
Exponential Data

My Notes

c. Scales may be easier to choose if only a subset of the data is graphed and if different scales are used for the horizontal and vertical axes. Determine an appropriate subset of the data and a scale for the graph. Plot the data and label and scale the axes. Draw a function that fits the plotted data.



One possible answer is to choose the subset {(1, 10), (2, 100), (3, 1000), (4, 10,000)} and plot those points.

d. Write a function $G(x)$ for the ground motion caused compared to a magnitude 0 earthquake by a magnitude x earthquake.
$G(x) = 10^x$

Check Your Understanding

4. What is the domain of the function in Item 3d? Is the graph of the function continuous?

5. Use the graph from Item 3c to estimate how many times greater the ground motion of an earthquake of magnitude 3.5 is than a magnitude 0 earthquake. Solve the equation you wrote in Item 3d to check that your estimate is reasonable.

6. Make sense of problems. In Item 3, the data were plotted so that the ground motion caused by the earthquake was a function of the magnitude of the earthquake.

a. Is the ground motion a result of the magnitude of an earthquake, or is the magnitude of an earthquake the result of ground motion?
An earthquake's magnitude is not assigned until an earthquake actually happens, so an earthquake's magnitude is a result of ground motion.

b. Based your answer to part a, would you choose ground motion or magnitude as the independent variable of a function relating the two quantities? What would you choose as the dependent variable?
Ground motion should be the independent variable and magnitude should be the dependent variable of a function relating the two quantities.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand why the upper limit of the domain is 10 in the context of the problem.

Answers

4. [0, 10]; Yes, it is continuous.
5. Estimate is (3.5, 3000); actual is (3.5, 3162.278).

6–7 Close Reading, Create Representations, Debriefing

Students should realize that even though they were asked to plot ground motion as a function of the magnitude in Item 3d, in reality magnitude is a function of ground motion. An earthquake's magnitude is not assigned until an earthquake actually happens, so it does not make sense for the earthquake's ground motion to be a function of its magnitude.

Differentiating Instruction

To **support** students in reading problem scenarios, carefully group students to ensure that all students participate and have an opportunity for meaningful reading and discussion. Suggest that group members each read a sentence and explain what that sentence means to them. Group members can then confirm one another's understanding of the key information provided for the problem.

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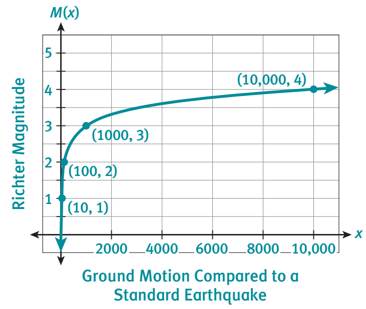
6–7 (continued) Close Reading, Create Representations, Debriefing
In Item 7a, the intent is that students consider each function that they graphed to fit the data over the entire domain and to begin to realize that the functions are inverses.

After completing Items 7a–c, students should be aware that the two functions are inverses. It is not expected that students be able to determine an algebraic rule for the function that is the inverse of $y = 10^x$.

My Notes

c. Make a new graph of the data plotted Item 3c so that the magnitude of the earthquake is a function of the ground motion caused by the earthquake. Scale the axes and draw a function that fits the plotted data.



7. Let the function you graphed in Item 6c be $y = M(x)$, where M is the magnitude of an earthquake for which there is x times as much ground motion as a magnitude 0 earthquake.

a. Identify a reasonable domain and range of the function $y = G(x)$ from Item 3d and the function $y = M(x)$ in this situation. Use interval notation.

	Domain	Range
$y = G(x)$	[0, 10]	(0, 10,000,000,000)
$y = M(x)$	(0, 10,000,000,000)	[0, 10]

b. In terms of the problem situation, describe the meaning of an ordered pair on the graphs of $y = G(x)$ and $y = M(x)$.

$y = G(x)$ Richter magnitude, Ground motion compared to magnitude 0 earthquake

$y = M(x)$ Ground motion compared to magnitude 0 earthquake, Richter magnitude

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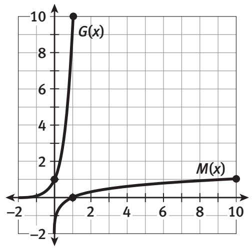
Lesson 22-1
Exponential Data

My Notes

c. A portion of the graphs of $y = G(x)$ and $y = M(x)$ is shown on the same set of axes. Describe any patterns you observe.



Sample answer: The two functions are symmetric about the line $y = x$. The values of x and y in $M(x)$ are the values of y and x in $G(x)$.

Check Your Understanding

8. How did you choose the scale of the graph you drew in Item 6c?

9. What is the relationship between the functions G and M?

LESSON 22-1 PRACTICE

How does the ground motion caused by earthquakes of these magnitudes compare?

10. magnitude 5.0 compared to magnitude 2.0

11. magnitude 7.0 compared to magnitude 0

12. magnitude 6.0 compared to magnitude 5.0

13. A 1933 California earthquake had a Richter scale reading of 6.3. How many times more powerful was the Alaska 1964 earthquake with a reading of 8.3?

14. Critique the reasoning of others. Garrett said that the ground motion of an earthquake of magnitude 6 is twice the ground motion of an earthquake of magnitude 3. Is Garrett correct? Explain.

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Check Your Understanding

Debrief students' answers to these items to ensure that they understand the functions are inverses. Have them list a few points from each function and reverse the coordinates to help them understand this.

Answers

8. reversed the axes from the graph for Item 3c

9. The functions G and M are inverse functions.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 22-1 PRACTICE

10. Magnitude 5.0 is 1000 times greater than magnitude 2.0.

11. Magnitude 7.0 is 10,000,000 times greater than magnitude 0.

12. Magnitude 6.0 is 10 times greater than magnitude 5.0.

13. 10,000

14. No; the function does not increase linearly. The ground motion of an earthquake of magnitude 6 is 1000 times the ground motion of an earthquake of magnitude 3.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand that the ground motion function is not linear. Have students refer back to the tables in Item 1 to confirm this.

Lesson 22-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2 #3–5

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find the inverse of each function.

1. $f(x) = -2x + 9$ $\left[f^{-1}(x) = \frac{9-x}{2}\right]$
2. $f(x) = x^2 - 4$ $\left[f^{-1}(x) = \sqrt{x+4}\right]$
3. $f(x) = x^3 + 1$ $\left[f^{-1}(x) = \sqrt[3]{x-1}\right]$

1 Quickwrite, Create Representations, Debriefing Encourage students to graph $y = \log x$ over various windows. A window of $[-7, 40]$ with a scale of 10 for x and $[-1, 2]$ with a scale of 1 for y gives a nice but limited view of the function. In part b, students can use the graph, tables, or the log function to evaluate $M(120{,}000)$. In part c, students can find the intersection of the graphs of $M(y) = \log x$ and $y = 7.9$. Alternatively, students could use the TABLE feature of the calculator to determine the answer. Students may need to appropriately rescale their graphs. Students should consult their graphing calculator manuals. For some students, writing this process in their notes will be helpful, as they can refer to it again and again as they work through the course. For additional technology resources, visit SpringBoard online.

Developing Math Language

Students should add the definition of *common logarithmic function* to their math notebooks. Discuss why the use of *common* is necessary when describing these types of functions. Students will learn about logarithms in other bases in Activity 23.

2 Create Representations, Debriefing Students numerically verify that $y = 10^x$ and $y = \log x$ are inverses of each other. Students may struggle with the last row of the table. Class discussion should reinforce the property of inverse functions, which states that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$. It is important that students know that $10^{\log x} = x$ and $\log 10^x = x$. Have them add these identities to their math notebooks.

ACTIVITY 22 *continued*

Lesson 22-2
The Common Logarithm Function

My Notes

Learning Targets:

- Use technology to graph $y = \log x$.
- Evaluate a logarithm using technology.
- Rewrite exponential equations as their corresponding logarithmic equations.
- Rewrite logarithmic equations as their corresponding exponential equations.

SUGGESTED LEARNING STRATEGIES: Close Reading, Vocabulary Organizer, Create Representations, Quickwrite, Think-Pair-Share

The Richter scale uses a base 10 ***logarithmic*** scale. A base 10 logarithmic scale means that when the ground motion is expressed as a power of 10, the magnitude of the earthquake is the exponent. You have seen this function $G(x) = 10^x$, where x is the magnitude, in Item 3d of the previous lesson.

The function M is the inverse of an exponential function G whose base is 10. The algebraic rule for M is a ***common logarithmic*** function. Write this function as $M(x) = \log x$, where x is the ground motion compared to a magnitude 0 earthquake.

MATH TERMS

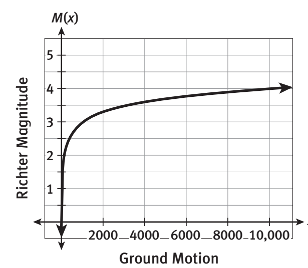
A **logarithm** is an exponent to which a base is raised that results in a specified value.

A **common logarithm** is a base 10 logarithm, such as $\log x$ • $\log 100 = 2$ because $10^2 = 100$.

TECHNOLOGY TIP

The LOG key on your calculator is for common, or base 10, logarithms.

1. Graph $M(x) = \log x$ on a graphing calculator.
   a. Make a sketch of the calculator graph. Be certain to label and scale each axis.



   b. Use M to estimate the magnitude of an earthquake that causes 120,000 times the ground motion of a magnitude 0 earthquake. Describe what would happen if this earthquake were centered beneath a large city.
   $M(120{,}000) \approx 5.08$. According to the information given in the previous lesson, an earthquake of this magnitude could cause damage to buildings.
   c. Use M to determine the amount of ground motion caused by the 2002 magnitude 7.9 Denali earthquake compared to a magnitude 0 earthquake.
   79,432,823.47 times as much motion as magnitude 0.

2. Complete the tables below to show the relationship between the exponential function base 10 and its inverse, the common logarithmic function.

x	$y = 10^x$
0	$10^0 = 1$
1	$10^1 = 10$
2	$10^2 = 100$
3	$10^3 = 1000$
$\log x$	$10^{\log x} = x$

x	$y = \log x$
$1 = 10^0$	$\log 1 = 0$
$10 = 10^1$	$\log (10) = 1$
$100 = 10^2$	$\log (100) = 2$
$1000 = 10^3$	$\log (1000) = 3$
10^x	$\log 10^x = x$

MATH TIP

You can also write the equation $y = \log x$ as $y = \log_{10} x$. In the equation $y = \log x$, 10 is understood to be the base. Just as exponential functions can have bases other than 10, ***logarithmic functions*** can also be expressed with bases other than 10.

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Lesson 22-2
The Common Logarithm Function

ACTIVITY 22
continued

3. Use the information in Item 2 to write a logarithmic statement for each exponential statement.
   a. $10^4 = 10{,}000$ $\log(10{,}000) = 4$
   b. $10^{-1} = \frac{1}{10}$ $\log\left(\frac{1}{10}\right) = -1$

4. Use the information in Item 2 to write each logarithmic statement as an exponential statement.
   a. $\log 100{,}000 = 5$ $10^5 = 100{,}000$
   b. $\log\left(\frac{1}{100}\right) = -2$ $10^{-2} = \frac{1}{100}$

5. Evaluate each logarithmic expression without using a calculator.
   a. $\log 1000$ 3
   b. $\log \frac{1}{10{,}000}$ -4

My Notes

MATH TIP

Recall that two functions are inverses when $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

The exponent x in the equation $y = 10^x$ is the common logarithm of y. This equation can be rewritten as $\log y = x$.

Check Your Understanding

6. What function has a graph that is symmetric to the graph of $y = \log x$ about the line $y = x$?
7. Evaluate $\log 10^x$ for $x = 1, 2, 3$, and 4.
8. Let $f(x) = 10^x$ and let $g(x) = f^{-1}(x)$. What is the algebraic rule for $g(x)$?

LESSON 22-2 PRACTICE

9. Evaluate without using a calculator.
   a. $\log 10^6$
   b. $\log 1{,}000{,}000$
   c. $\log \frac{1}{100}$
10. Write an exponential statement for each.
    a. $\log 10 = 1$
    b. $\log \frac{1}{1{,}000{,}000} = -6$
    c. $\log a = b$
11. Write a logarithmic statement for each.
    a. $10^7 = 10{,}000{,}000$
    b. $10^0 = 1$
    c. $10^m = n$
12. **Model with mathematics.** The number of decibels D of a sound is modeled with the equation $D = 10\log\left(\frac{I}{10^{-12}}\right)$ where I is the intensity of the sound measured in watts. Find the number of decibels in each of the following:
    a. whisper with $I = 10^{-10}$
    b. normal conversation with $I = 10^{-6}$
    c. vacuum cleaner with $I = 10^{-4}$
    d. front row of a rock concert with $I = 10^{-1}$
    e. military jet takeoff with $I = 10^2$

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3–5 Create Representations, Think-Pair-Share, Debriefing
Students should have noticed the relationship between exponentials and logarithms. Additional guidance is provided in the Math Tip.

If students have trouble with Item 5, remind them that a common logarithm is simply an "exponent" for base 10. To evaluate log 1000, find the exponent to which 10 should be raised in order to get 1000.

Check Your Understanding
Debrief students' answers to these items to ensure that they understand that the inverse of an exponential function is a logarithmic function and the inverse of a logarithmic function is an exponential function.

Answers
6. $y = 10^x$
7. 1, 2, 3, 4
8. $g(x) = \log x$

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 22-2 PRACTICE
9. a. 6
   b. 6
   c. -2
10. a. $10^1 = 10$
    b. $10^{-6} = \frac{1}{1{,}000{,}000}$
    c. $10^b = a$
11. a. $\log 10{,}000{,}000 = 7$
    b. $\log 1 = 0$
    c. $\log n = m$
12. a. 20
    b. 60
    c. 80
    d. 110
    e. 140

ADAPT

Check students' answers to the Lesson Practice to ensure that they can switch between exponential and logarithmic form. Allow students to confirm their answers to Items 9 and 12 using a scientific calculator.

Lesson 22-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–2 #3–4 #5–6
Check Your Understanding
#9–11
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to evaluate each expression.

1. $2^3 \cdot 2^{-1}$ [4]
2. $\left(3^{-2}\right)^{-2}$ [81]
3. $\dfrac{4^5}{\left(2^2\right)^3}$ [16]

Teacher to Teacher

In this lesson and the next, students will discover that the product rule, the quotient rule, and the power rule hold true for common logarithms. In Activity 23, students will learn that these properties extend to other logarithms of other bases as well.

Differentiating Instruction

Students need to be familiar with the properties of exponents before continuing with this activity. Review these properties with students.

1–2 Create Representations, Activating Prior Knowledge, Debriefing Item 1 prompts students for information that they should have learned previously. The information, in turn, will lead them to make conjectures about the connections between these properties and the properties of logarithms.

Students will use data from the tables in Item 2 in items that follow. It is important to see that students have correct answers for Items 1 and 2 before they proceed to the next items.

3–4 Create Representations If students do not recognize a property in Item 3, the next items will guide them to the property.

My Notes

Learning Targets:

- Make conjectures about properties of logarithms.
- Write and apply the Product Property and Quotient Property of Logarithms.
- Rewrite logarithmic expressions by using properties.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Look for a Pattern, Quickwrite, Guess and Check

You have already learned the properties of exponents. Logarithms also have properties.

1. Complete these three properties of exponents.

$a^m \cdot a^n =$ a^{m+n}

$\dfrac{a^m}{a^n} =$ a^{m-n}

$(a^m)^n =$ $a^{m \cdot n}$

2. **Use appropriate tools strategically.** Use a calculator to complete the tables below. Round each answer to the nearest thousandth.

x	$y = \log x$
1	0
2	0.301
3	0.477
4	0.602
5	0.699

x	$y = \log x$
6	0.778
7	0.845
8	0.903
9	0.954
10	1

3. Add the logarithms from the tables in Item 2 to see if you can develop a property. Find each sum and round each answer to the nearest thousandth.

$\log 2 + \log 3 =$ 0.778 = log 6

$\log 2 + \log 4 =$ 0.903 = log 8

$\log 2 + \log 5 =$ 1 = log 10

$\log 3 + \log 3 =$ 0.954 = log 9

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MINI-LESSON: Review of Exponent Properties

Students need to be familiar with the properties of exponents in this lesson. If students need to review these properties, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

Lesson 22-3
Properties of Logarithms

My Notes

4. Compare the answers in Item 3 to the tables of data in Item 2.

a. Express regularity in repeated reasoning. Is there a pattern or property when these logarithms are added? If yes, explain the pattern that you have found.

When two logarithms of like bases are added together, the result is the logarithm of the product of the input values.

b. State the property of logarithms that you found by completing the following statement.

$\log m + \log n =$ ___**log (*mn*)**___

5. Explain the connection between the property of logarithms stated in Item 4 and the corresponding property of exponents in Item 1.

When exponential expressions with like bases are multiplied, the exponents are added. When logarithms with like bases are added, the result is the logarithm of the product of their inputs.

6. Graph $y_1 = \log 2 + \log x$ and $y_2 = \log 2x$ on a graphing calculator. What do you observe? Explain.

The two functions are identical because $y_2 = \log(2x) = \log(2) + \log(x) = y_1$ by the property in Item 4b.

TECHNOLOGY TIP

When using the LOG key on a graphing calculator, a leading parenthesis is automatically inserted. The closing parenthesis for logarithmic expressions must be entered manually. So entering $\log 2 + \log x$ without closing the parentheses will NOT give the correct result.

Check Your Understanding

Identify each statement as true or false. Justify your answers.

7. $\log mn = (\log m)(\log n)$

8. $\log xy = \log x + \log y$

9. Make a conjecture about the property of logarithms that relates to the property of exponential equations that states the following:

$$\frac{a^m}{a^n} = a^{m-n}.$$

The conjecture is $\log(m) - \log(n) = \log\left(\frac{m}{n}\right)$.

10. Use the information from the tables in Item 2 to provide examples that support your conjecture in Item 9.

Sample answers:

$\log(4) - \log(2) = 0.301 = \log\left(\frac{4}{2}\right) = \log(2)$

$\log(6) - \log(2) = 0.477 = \log\left(\frac{6}{2}\right) = \log(3)$

11. Graph $y_1 = \log x - \log 2$ and $y_2 = \log \frac{x}{2}$ on a graphing calculator. What do you observe?

The graphs of the two functions are identical.

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3–4 (continued) Look for a Pattern, Debriefing Students should realize that when two logarithms with like bases are added, the result is the logarithm of the product of the input values.

5–6 Quickwrite, Create Representations, Debriefing Students may struggle with making the connection in Item 5. When exponential expressions with like bases are multiplied, the exponents are added. When logarithms with like bases are added, the result is the logarithm of the product of their inputs. Help students to see that logarithms are exponents that have been expressed using different notation.

Students have looked at the property in Item 6 both numerically and analytically in terms of the corresponding property of exponents. Now they will validate the property graphically. A suggested graphing window would be [0, 10] for the x-axis and [−2, 2] for the y-axis.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the properties of logarithms presented in this lesson.

Answers

7. False; it is the sum of the logs: $\log m + \log n$.

8. true by the property in Item 4b

9–11 Guess and Check, Create Representations, Look for a Pattern, Debriefing Students will make a conjecture or guess in Item 9 and then validate or check it in the items that follow. Students who struggle with Item 9 should be directed to take a similar approach to what they did in Item 3 and Item 4. If students continue to have difficulty with this concept, have them answer Item 11 and then return to Item 9.

In Items 10 and 11, students validate their conjecture both numerically and graphically. A suggested graphing window would be [0, 10] for the x-axis and [−2, 2] for the y-axis.

Check Your Understanding

Debrief students' answers to these items to ensure that they can use the Product and Quotient Properties to rewrite logarithms.

Answers

12. Sample answers: $\log (9 \cdot 4) = \log 9 + \log 4 = 0.954 + 0.602 = 1.556$;
$\log (6 \cdot 6) = \log 6 + \log 6 = 0.778 + 0.778 = 1.556$

13. Sample answers: $\log\left(\frac{10}{5}\right) = \log 10 - \log 5 = 1 - 0.699 = 0.301$

14. $\log 7 = 0.845$; $\log 3 = 0.477$; $\log 4 = 0.602$; $0.477 + 0.602 = 1.079$, not 0.845.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 22-3 PRACTICE

15. a. $\log\left(\frac{8}{3}\right) = \log 8 - \log 3 = 0.903 - 0.477 = 0.426$

b. $\log 24 = \log (8 \cdot 3) = \log 8 + \log 3 = 0.903 + 0.477 = 1.38$

c. $\log 64 = \log (8 \cdot 8) = \log 8 + \log 8 = 0.903 + 0.903 = 1.806$

d. $\log 27 = \log (3 \cdot 3 \cdot 3) = \log 3 + \log 3 + \log 3 = 0.477 + 0.477 + 0.477 = 1.431$

16. a. $\log\left(\frac{4}{9}\right) = \log 4 - \log 9 = 0.602 - 0.954 = -0.352$

b. $\log 2.25 = \log \frac{9}{4} = \log 9 - \log 4 = 0.954 - 0.602 = 0.352$

c. $\log 144 = \log (4 \cdot 4 \cdot 9) = \log 4 + \log 4 + \log 9 = 0.602 + 0.602 + 0.954 = 2.158$

d. $\log 81 = \log (9 \cdot 9) = \log 9 + \log 9 = 0.954 + 0.954 = 1.908$

17. $\log\left(\frac{7x}{3y}\right)$

18. $\log 8 + \log m - (\log 9 + \log n)$

19. $\log\left(\frac{8 \cdot 2}{4}\right) = \log 4 = 0.602$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to evaluate a logarithmic expression and rewrite an expression as a single logarithm. As an additional activity, use index cards to create a game for students to match expressions. For example, $\log\left(\frac{4}{9}\right)$ would match the expression $\log 4 - \log 9$.

Check Your Understanding

Use the information from the tables in Item 2 and the properties in Items 4b and 9.

12. Write two different logarithmic expressions to find a value for log 36.

13. Write a logarithmic expression that contains a quotient and simplifies to 0.301.

14. **Construct viable arguments.** Show that $\log (3 + 4) \neq \log 3 + \log 4$.

LESSON 22-3 PRACTICE

Use the table of logarithmic values at the beginning of the lesson to evaluate the logarithms in Items 15 and 16. Do not use a calculator.

15. a. $\log\left(\frac{8}{3}\right)$

b. log 24

c. log 64

d. log 27

16. a. $\log\left(\frac{4}{9}\right)$

b. log 2.25

c. log 144

d. log 81

17. Rewrite $\log 7 + \log x - (\log 3 + \log y)$ as a single logarithm.

18. Rewrite $\log\left(\frac{8m}{9n}\right)$ as a sum of four logarithmic terms.

19. **Make use of structure.** Rewrite $\log 8 + \log 2 - \log 4$ as a single logarithm and evaluate the result using the table at the beginning of the lesson.

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Lesson 22-4
More Properties of Logarithms

ACTIVITY 22 *continued*

My Notes

Learning Targets:
- Make conjectures about properties of logarithms.
- Write and apply the Power Property of Logarithms.
- Rewrite logarithmic expressions by using their properties.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations

1. Make a conjecture about the property of logarithms that relates to the property of exponents that states the following: $(a^m)^n = a^{mn}$.
$\log (m^n) = n \log (m)$

2. Use the information from the tables in Item 2 in the previous lesson and the properties developed in Items 4 and 9 in the previous lesson to support your conjecture in Item 1.
Possible answers are given below.
$\log (3^2) = \log (3 \cdot 3) = \log (3) + \log (3) = 2 \log (3)$
$\log (4^2) = \log (4 \cdot 4) = \log (4) + \log (4) = 2 \log (4)$
$\log (2^3) = \log (2 \cdot 2 \cdot 2) = \log (2) + \log (2) + \log (2) = 3 \log (2)$

3. Use appropriate tools strategically. Graph $y_1 = 2 \log x$ and $y_2 = \log x^2$ on a graphing calculator. What do you observe?
The graphs of the two functions are identical.

Check Your Understanding

Identify each statement as true or false. Justify your answer.

4. $2 \log \sqrt{m} = \log m$

5. $\log 10^2 = \log 2^{10}$

6. Express regularity in repeated reasoning. The logarithmic properties that you conjectured and then verified in this lesson and the previous lesson are listed below. State each property.

Product Property: $\log m + \log n = \log (mn)$

Quotient Property: $\log(m) - \log(n) = \log\left(\frac{m}{n}\right)$

Power Property: $\log (m^n) = n \log (m)$

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6–9 Think-Pair-Share, Create Representations, Debriefing Item 4 gives students an opportunity to reflect on the properties derived in the activity. After sharing answers with the entire group to make sure all responses are correct, students can record the information in their math notebooks.

Monitor group discussions to ensure that all members are participating. Pair or group students carefully to facilitate discussions and understanding of both routine language and mathematical terms.

Lesson 22-4

PLAN

Pacing: 1 class period

Chunking the Lesson
#1–3
Check Your Understanding
#6–9
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity
Ask students to identify each statement as true or false. If the statement is false, have them correct it.

1. $\log 3 + \log 5 = \log 8$ [*false; $\log 3 + \log 5 = \log 15$*]
2. $\log 10 - \log 5 = \log 2$ [*true*]
3. $\log 1000 = 3$ [*true*]

1–3 Guess and Check, Create Representations, Look for a Pattern, Debriefing Students may need coaching to develop the property in Item 1. There are two ways to approach this problem, following the patterns from Items 2–11. Students could try either $\log (u)^v$ or $\log (u^v)$. They may struggle with $\log (u)^v$ as there is no pattern to be found. However, if students try $\log (u^v)$, they will have more success if they recognize exponentiation as repeated multiplication and apply the Product Property of Logarithms:

$$\begin{aligned}\log(u^v) &= \log(u \cdot u \cdot u \cdot \ldots \cdot u)\\ &= \log(u) + \log(u) + \ldots + \log(u)\\ &= v \log(u)\end{aligned}$$

In Item 2, students should apply the Product Property of Logarithms to verify their conjecture numerically. For example,

$$\begin{aligned}\log(3^2) &= \log(3 \cdot 3)\\ &= \log 3 + \log 3\\ &= 2 \log 3\end{aligned}$$

In Item 3, students verify their conjecture graphically. A suggested graphing window would be [0, 10] for the x-axis and [−2, 2] for the y-axis.

Check Your Understanding
Debrief students' answers to these items to ensure that they can correctly use properties of logarithms to justify their answers.

Answers
4. True; by the property in Item 1, $2 \log \sqrt{m} = \log \sqrt{m}^2 = \log m$.
5. False; $\log 10^2 = 2$

6–9 (continued) Items 7–9 give students an opportunity to apply the logarithm properties. They also provide an opportunity to assess student understanding.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand that the log of a negative number is not a real number.

Answers

10. The log of a sum does not equal the sum of the logs; only the product equals the sum of the logs.
11. $\log(-100)$ is not defined since it represents the exponent you would raise 10 to in order to get -100. A base of 10 will never give a negative value.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 22-4 PRACTICE

12. $\log 100 = 2$
13. $\log \frac{1}{10} = -1$
14. $\log 10{,}000 = 4$
15. $\log \frac{1}{100} = -2$
16. $\log 100\left(\frac{1}{100}\right) = \log 1 = 0$
17. $\log b + 3\log c + 2\log d$

ADAPT

Check students' answers to the Lesson Practice to ensure that they can both rewrite logarithmic expressions as a single logarithm and expand a single logarithm into an expression. Students should easily and fluently go back and forth between the two forms. Allow students to use a calculator to check their work.

My Notes

7. Use the properties from Item 6 to rewrite each expression as a single logarithm. Assume all variables are positive.
   a. $\log x - \log 7$
      $\log \frac{x}{7}$
   b. $2\log x + \log y$
      $\log(x^2y)$

8. Use the properties from Item 6 to expand each expression. Assume all variables are positive.
   a. $\log 5xy^4$
      $\log 5 + \log x + 4\log y$
   b. $\log \frac{x}{y^3}$
      $\log x - 3\log y$

9. Rewrite each expression as a single logarithm. Then evaluate.
   a. $\log 2 + \log 5$
      $\log 10 = 1$
   b. $\log 5000 - \log 5$
      $\log 1000 = 3$
   c. $2\log 5 + \log 4$
      $\log 100 = 2$

Check Your Understanding

10. Explain why $\log(a + 10)$ does not equal $\log a + 1$.
11. Explain why $\log(-100)$ is not defined.

LESSON 22-4 PRACTICE

Attend to precision. Rewrite each expression as a single logarithm. Then evaluate the expression without using a calculator.

12. $\log 5 + \log 20$
13. $\log 3 - \log 30$
14. $2\log 400 - \log 16$
15. $\log \frac{1}{400} + 2\log 2$
16. $\log 100 + \log\left(\frac{1}{100}\right)$.
17. Expand the expression $\log bc^3d^2$.

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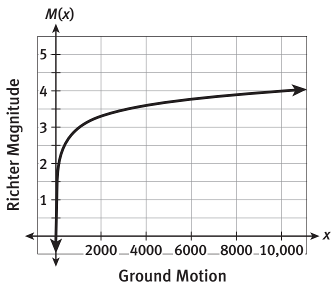
ACTIVITY 22 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 22-1



1. What is the y-intercept of the graph?

2. What is the x-intercept of the graph?

3. Is $M(x)$ an increasing or decreasing function?

4. Which of these statements are NOT true regarding the graph above?
A. The graph contains the point (1, 0).
B. The graph contains the point (10, 1).
C. The domain is $x > 0$.
D. The x-axis is an asymptote.

Lesson 22-2

5. Use a calculator to find a decimal approximation rounded to three decimal places.
a. $\log 47$
b. $\log 32.013$
c. $\log\left(\frac{5}{7}\right)$
d. $\log -20$

6. A logarithm is a(n)
A. variable.
B. constant.
C. exponent.
D. coefficient.

7. Write an exponential statement for each logarithmic statement below.
a. $\log 10{,}000 = 4$
b. $\log \frac{1}{1{,}000{,}000{,}000} = -9$
c. $\log a = 6$

8. Write a logarithmic statement for each exponential statement below.
a. $10^{-2} = \frac{1}{100}$
b. $10^1 = 10$
c. $10^4 = n$

9. Evaluate without using a calculator.
a. $\log 10^5$
b. $\log 100$
c. $\log \frac{1}{100{,}000}$

10. If $\log a = x$, and $10 < a < 100$, what values are acceptable for x?
A. $0 < x < 1$
B. $1 < x < 2$
C. $2 < x < 3$
D. $10 < x < 100$

Lesson 22-3

11. If $\log 2 = 0.301$ and $\log 3 = 0.447$, find each of the following using only these values and the properties of logarithms. Show your work.
a. $\log 6$
b. $\log\left(\frac{2}{3}\right)$
c. $\log 1.5$
d. $\log 18$

ACTIVITY 22 Continued

ACTIVITY PRACTICE

1. There is no y-intercept.
2. (1, 0)
3. $M(x)$ is increasing.
4. D
5. a. 1.672
b. 1.505
c. −0.146
d. not a real number
6. C
7. a. $10^4 = 10{,}000$
b. $10^{-9} = \frac{1}{1{,}000{,}000{,}000}$
c. $10^6 = a$
8. a. $\log \frac{1}{100} = -2$
b. $\log 10 = 1$
c. $\log n = 4$
9. a. 5
b. 2
c. −5
10. B
11. a. $0.301 + 0.447 = 0.748$
b. $0.301 - 0.447 = -0.146$
c. $0.447 - 0.301 = 0.146$
d. $0.301 + 0.447 + 0.447 = 1.195$

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12. B
13. Sample answer: Simplify $\log 10^3 + \log 10^5 = \log 10^8$ to $3 \log 10 + 5 \log 10 = 8 \log 10$. Since $\log 10 = 1$, the log equation becomes $3 + 5 = 8$, which is precisely the same operation used in the exponent product.
14. a. $\log \frac{2x}{3y}$
    b. $\log \frac{5}{7}$
    c. $\log \frac{24 \cdot 12}{6} = \log 48$
15. a. $\log 3 + \log x - (\log 8 + \log y)$
    b. $\log (m + v) - \log 3$
    c. $\log 4 - \log (9 - u)$
16. a. 0.602
    b. 1.431
    c. 0.151
    d. 0.540
17. a. $\log uv = \log u + \log v$
    b. $\log \frac{u}{v} = \log (u) - \log (v) =$
    c. $\log u^v = v \log (u)$
18. a. $\log 1000 = 3$
    b. $\log 1 = 0$
    c. $\log 10 = 1$
19. a. $\log x + 2 \log y$
    b. $\log x + \log y - \log z$
    c. $3 \log a + 2 \log b$
20. a. 3.816
    b. 2.709
    c. 1.857
21. a. $2\, m \log n$
    b. 0
    c. $7 \log 2$
22. D
23. $\log 10^x - \log 10^4$
    $= x \log 10 - 4 \log 10$
    $= (x - 4) \log 10$
    $= (x - 4)(1)$
    $= x - 4$
    $\log 10^\pi - \log 10^4$
    $= 3.14 - 4$
    $= -0.86$

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

ACTIVITY 22 *continued*

Logarithms and Their Properties

Earthquakes and Richter Scale

12. Which expression does NOT equal 3?
  A. $\log 10^3$
  B. $\dfrac{\log 10^5}{\log 10^2}$
  C. $\log\left(\dfrac{10^7}{10^4}\right)$
  D. $\log 10^4 - \log 10$

13. Explain the connection between the exponential equation ($10^3 \cdot 10^5 = 10^8$) and the logarithmic equation ($\log 10^3 + \log 10^5 = \log 10^8$).

14. Rewrite each expression as a single logarithm.
  a. $\log 2 + \log x - (\log 3 + \log y)$
  b. $\log 5 - \log 7$
  c. $(\log 24 + \log 12) - \log 6$

15. Expand each expression.
  a. $\log\left(\dfrac{3x}{8y}\right)$
  b. $\log\left(\dfrac{m+v}{3}\right)$
  c. $\log\left(\dfrac{4}{9-u}\right)$

16. If $\log 2 = 0.301$ and $\log 3 = 0.477$, find each of the following using the properties of logarithms.
  a. $\log 4$
  b. $\log 27$
  c. $\log \sqrt{2}$
  d. $\log \sqrt{12}$

Lesson 22-4

17. Complete each statement to illustrate a property for logarithms.
  a. Product Property $\log uv = ?$
  b. Quotient Property $\log \frac{u}{v} = ?$
  c. Power Property $\log u^v = ?$

18. Rewrite each expression as a single logarithm. Then evaluate without using a calculator.
  a. $\log 500 + \log 2$
  b. $2 \log 3 + \log \frac{1}{9}$
  c. $\log 80 - 3 \log 2$

19. Expand each expression.
  a. $\log xy^2$
  b. $\log \dfrac{xy}{z}$
  c. $\log a^3b^2$

20. If $\log 8 = 0.903$ and $\log 3 = 0.477$, find each of the following using the properties of logarithms.
  a. $\log 3^8$
  b. $\log (2^3)^3$
  c. $\log 8(3^2)$

21. Write each expression without using exponents.
  a. $m \log n + \log n^m$
  b. $\log (mn)^0$
  c. $\log 2^4 + \log 2^3$

22. Which of the following statements is TRUE?
  A. $\log \dfrac{x}{y} = \dfrac{\log x}{\log y}$
  B. $\log \dfrac{x}{y} = y \log x$
  C. $\log (x + y) = \log x + \log y$
  D. $\log \sqrt{x} = \frac{1}{2} \log x$

MATHEMATICAL PRACTICES

Reason Abstractly and Quantitatively

23. Verify using the properties of logarithms that $\log 10^x - \log 10^4 = x - 4$. Then evaluate for $x = \pi$, using 3.14 for π.

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Exponential Functions and Common Logarithms
WHETHER OR NOT

Embedded Assessment 2
Use after Activity 22

1. **Reason quantitatively.** Tell whether or not each table contains data that can be modeled by an exponential function. Provide an equation to show the relationship between x and y for the sets of data that are exponential.

   a.

x	0	1	2	3
y	3	6	12	24

   b.

x	0	1	2	3
y	2	4	6	8

   c.

x	0	1	2	3
y	108	36	12	4

2. Tell whether or not each function is increasing. State *increasing* or *decreasing*, and give the domain, range, and y-intercept of the function.

   a. $y = 4\left(\frac{2}{3}\right)^x$ b. $y = -3(4)^x$

3. Let $g(x) = 2(4)^{x+3} - 5$.
   a. Describe the function as a transformation of $f(x) = 4^x$.
   b. Graph the function using your knowledge of transformations.
   c. What is the horizontal asymptote of the graph of g?

4. Rewrite each exponential equation as a common logarithmic equation.

   a. $10^3 = 1000$ b. $10^{-4} = \frac{1}{10,000}$ c. $10^7 = 10,000,000$

5. **Make use of structure.** Rewrite each common logarithmic equation as an exponential equation.

   a. $\log 100 = 2$ b. $\log 100,000 = 5$ c. $\log \frac{1}{100,000} = -5$

6. Evaluate each expression without using a calculator.

   a. $\log 1000$ b. $\log 1$ c. $\log 2 + \log 50$

7. Evaluate using a calculator. Then rewrite each expression as a single logarithm without exponents and evaluate again as a check.

   a. $\log 5 + \log 3$ b. $\log 3^4$ c. $\log 3 - \log 9$

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Common Core State Standards for Embedded Assessment 2

HSF-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

HSF-IF.B.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

HSF-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

HSF-IF.C.7e Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

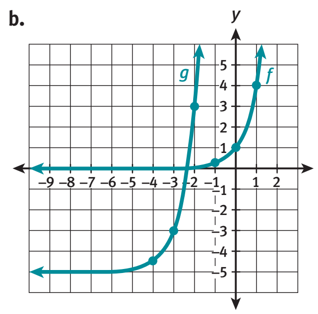
Embedded Assessment 2

Assessment Focus

- Examining exponential patterns and functions
- Identifying and analyzing exponential graphs
- Transforming exponential functions
- Graphing and transforming natural base exponential functions
- Examining common logarithmic functions
- Understanding properties of logarithms

Answer Key

1. a. exponential; $y = 3(2)^x$
   b. not exponential
   c. exponential; $y = 108\left(\frac{1}{3}\right)^x$
2. a. decreasing; domain: $(-\infty, \infty)$; range: $(0, \infty)$; y-intercept: 4
   b. decreasing; domain: $(-\infty, \infty)$; range: $(-\infty, 0)$; y-intercept: -3
3. a. To obtain the graph of g, vertically stretch the graph of f by a factor of 2, horizontally translate to the left 3 units, and then vertically translate down 5 units.
   b.



   c. $y = -5$
4. a. $\log 1000 = 3$
   b. $\log \frac{1}{10,000} = -4$
   c. $\log 10,000,000 = 7$
5. a. $10^2 = 100$
   b. $10^5 = 10,000$
   c. $10^{-5} = \frac{1}{10,000}$
6. a. 3
   b. 0
   c. 2
7. a. log 15; 1.76091259
   b. 4 log 3; 1.908485019
   c. $\log \frac{1}{3}$; -0.4771212547

Teacher to Teacher

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Unpacking Embedded Assessment 3

Once students have completed this Embedded Assessment, turn to Embedded Assessment 3 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 3.

Embedded Assessment 2
Use after Activity 22

Exponential Functions and Common Logarithms

WHETHER OR NOT

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 2, 3c, 4–7)	• Clear and accurate understanding of how to determine whether a table of data represents an exponential function • Clear and accurate understanding of the features of exponential functions and their graphs including domain and range • Fluency in evaluating and rewriting exponential and logarithmic equations and expressions	• Largely correct understanding of how to determine whether a table of data represents an exponential function • Largely correct understanding of the features of exponential functions and their graphs including domain and range • Little difficulty when evaluating and rewriting exponential and logarithmic equations and expressions	• Partial understanding of how to determine whether a table of data represents an exponential function • Partial understanding of the features of exponential functions and their graphs including domain and range • Some difficulty when evaluating and rewriting logarithmic and exponential equations and expressions	• Little or no understanding of how to determine whether a table of data represents an exponential function • Inaccurate or incomplete understanding of the features of exponential functions and their graphs including domain and range • Significant difficulty when evaluating and rewriting logarithmic and exponential equations and expressions
Problem Solving (Item 1)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 3b)	• Fluency in recognizing exponential data and modeling it with an equation • Effective understanding of how to graph an exponential function using transformations	• Little difficulty in accurately recognizing exponential data and modeling it with an equation • Largely correct understanding of how to graph an exponential function using transformations	• Some difficulty with recognizing exponential data and modeling it with an equation • Partial understanding of how to graph an exponential function using transformations	• Significant difficulty with recognizing exponential data and model it with an equation • Mostly inaccurate or incomplete understanding of how to graph an exponential function using transformations
Reasoning and Communication (Items 1a, 3a)	• Clear and accurate justification of whether or not data represented an exponential model • Precise use of appropriate math terms and language to describe a function as a transformation of another function	• Adequate justification of whether or not data represented an exponential model • Adequate and correct description of a function as a transformation of another function	• Misleading or confusing justification of whether or not data represented an exponential model • Misleading or confusing description of a function as a transformation of another function	• Incomplete or inadequate justification of whether or not data represented an exponential model • Incomplete or mostly inaccurate description of a function as a transformation of another function

Common Core State Standards for Embedded Assessment 2 *(cont.)*

HSF-BF.A.1 Write a function that describes a relationship between two quantities.★

HSF-BF.A.1a Determine an explicit expression, a recursive process, or steps for calculation from a context.

HSF-BF.B.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

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Inverse Functions: Exponential and Logarithmic Functions

ACTIVITY 23

Undoing It All
Lesson 23-1 Logarithms in Other Bases

My Notes

Learning Targets:

- Use composition to verify two functions as inverse.
- Define the logarithm of *y* with base *b*.
- Write the Inverse Properties for logarithms.

SUGGESTED LEARNING STRATEGIES: Close Reading, Create Representations

In the first unit, you studied inverses of linear functions. Recall that two functions *f* and *g* are *inverses* of each other if and only if $f(g(x)) = x$ for all *x* in the domain of *g*, and $g(f(x)) = x$ for all *x* in the domain of *f*.

MATH TIP

To find the inverse of a function algebraically, interchange the *x* and *y* variables and then solve for *y*.

1. Find the inverse function $g(x)$ of the function $f(x) = 2x + 1$. Show your work.

   $y = 2x + 1$
   $x = 2y + 1$
   $y = \frac{x-1}{2}$
   $g(x) = \frac{x-1}{2}$

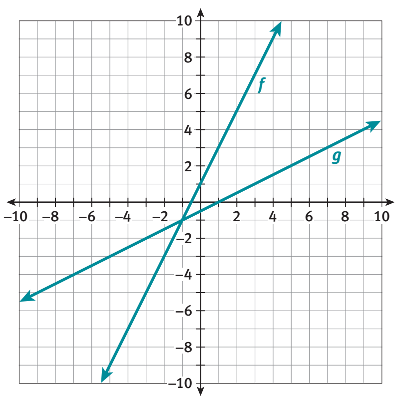
2. Use the definition of inverse functions to prove that $f(x) = 2x + 1$ and the $g(x)$ function you found in Item 1 are inverse functions.

   $f(g(x)) = 2\left(\frac{x-1}{2}\right) + 1 = x - 1 + 1 = x$

   $g(f(x)) = \frac{(2x+1)-1}{2} = \frac{2x}{2} = x$

3. Graph $f(x) = 2x + 1$ and its inverse $g(x)$ on the grid below. What is the line of symmetry between the graphs?

   The line of symmetry is $y = x$.



In a previous activity, you investigated exponential functions with a base of 10 and their inverse functions, the common logarithmic functions. Recall in the Richter scale situation that $G(x) = 10^x$, where *x* is the magnitude of an earthquake. The inverse function is $M(x) = \log x$, where *x* is the ground motion compared to a magnitude 0 earthquake.

Common Core State Standards for Activity 23

HSF-BF.A.1	Write a function that describes a relationship between two quantities.*
HSF-BF.A.1c (+)	Compose functions.
HSF-BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of *k* (both positive and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
HSF-BF.B.4	Find inverse functions.
HSF-BF.B.4b (+)	Verify by composition that one function is the inverse of another.

ACTIVITY 23
Investigative

Activity Standards Focus

In Activity 23, students extend the concept of logarithms to bases other than 10. They also extend their knowledge of inverse functions to include the inverse relationship between $y = b^x$ and $y = \log_b x$. Students will discover and apply properties of logarithms and apply the concept of graphing by transformations to logarithmic functions.

Lesson 23-1

PLAN

Pacing: 1 class period

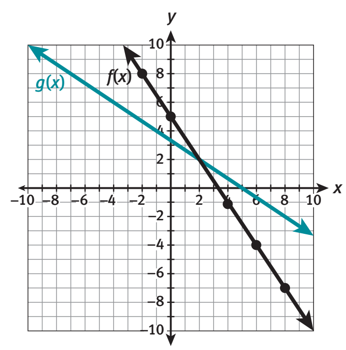
Chunking the Lesson

#1–3 #4 #5–6
#7–8
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Present the following graph of $f(x)$, and ask students to sketch the graphs of $f(x)$ and $g(x)$, the reflection of $f(x)$, over the line $y = x$. Have students use the graph to find $g(-1)$, $g(2)$, and $g(5)$.



$[g(-1) = 4, g(2) = 2, g(5) = 0]$

1–3 Activating Prior Knowledge, Create Representations These items review prior work with finding the inverse of a linear function and then graphing a function and its inverse. This then becomes the foundation for working with inverses of logarithmic functions.

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4 Activating Prior Knowledge, Create Representations This item reviews the concept of inverses that students investigated in Activity 22. Students see the equations and graphs of the common logarithmic function and its inverse, the general exponential function. This item provides the background as students begin to study logarithms with bases other than 10.

Developing Math Language

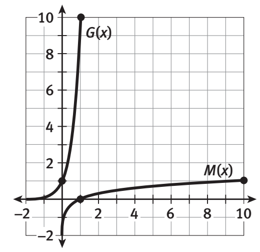
Here students are formally introduced to the logarithm of y base b and to logarithms with base e, or natural logarithms. Have students add the definitions and notations to their math noteooks. Point out that the base in a logarithmic function is the same as the base in the inverse exponential function.

5–6 Create Representations Students realize that exponential functions and logarithmic functions with the same base are inverses. Be aware that students may correctly answer Item 5 without fully understanding what it means for $y = b^x$ and $y = \log_b x$ to be inverse functions. Item 6 will help students with that understanding.

My Notes

4. A part of each of the graphs of $y = G(x)$ and $y = M(x)$ is shown below. What is the line of symmetry between the graphs? How does that line compare with the line of symmetry in Item 3?

The line of symmetry is $y = x$. It is the same as the line of symmetry in Item 3.



Logarithms with bases other than 10 have the same properties as common logarithms.

> The logarithm of y with base b, where $y > 0$, $b > 0$, $b \neq 1$, is defined as: $\log_b y = x$ if and only if $y = b^x$.

The exponential function $y = b^x$ and the logarithmic function $y = \log_b x$, where $b > 0$ and $b \neq 1$, are inverse functions.

MATH TIP

The notation f^{-1} is used to indicate the inverse of the function f.

5. Let $g(x) = f^{-1}(x)$, the inverse of function f. Write the rule for g for each function f given below.

a. $f(x) = 5^x$ — **$g(x) = \log_5 x$**

b. $f(x) = \log_4 x$ — **$g(x) = 4^x$**

c. $f(x) = \log_e x$ — **$g(x) = e^x$**

Logarithms with base e are called ***natural logarithms***, and "$\log_e$" is written **ln**. So, $\log_e x$ is written $\ln x$.

6. Use the functions from Item 5. Complete the expression for each composition.

a. $f(x) = 5^x$

$f(g(x)) =$ **$5^{\log_5 x}$** $= x$

$g(f(x)) =$ **$\log_5 5^x$** $= x$

b. $f(x) = \log_4 x$

$f(g(x)) =$ **$\log_4 4^x$** $= x$

$g(f(x)) =$ **$4^{\log_4 x}$** $= x$

c. $f(x) = e^x$

$f(g(x)) =$ **$e^{\ln x}$** $= x$

$g(f(x)) =$ **$\ln e^x$** $= x$

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Common Core State Standards for Activity 23 *(continued)*

HSF-IF.C.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
HSF-IF.C.7e	Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

Lesson 23-1
Logarithms in Other Bases

My Notes

7. Use what you learned in Item 6 to complete these *Inverse Properties of Logarithms*. Assume $b > 0$ and $b \neq 1$.

a. $b^{\log_b x} =$ ___x___ **b.** $\log_b b^x =$ ___x___

8. Simplify each expression.

a. $6^{\log_6 x}$ x **b.** $\log_3 3^x$ x

c. $7^{\log_7 x}$ x **d.** $\log 10^x$ x

e. $\ln e^x$ x **f.** $e^{\ln x}$ x

Check Your Understanding

9. Describe the process you use to find the inverse function $g(x)$ if $f(x) = 7x + 8$.

10. Construct viable arguments. Look at the graphs in Items 3 and 4. What can you conclude about the line of symmetry for a function and its inverse?

11. Answer each of the following as true or false. If false, explain your reasoning.

a. The "-1" in function notation f^{-1} means $\frac{1}{f}$.

b. Exponential functions are the inverse of logarithmic functions.

c. If the inverse is a function, then the original must be a function.

LESSON 23-1 PRACTICE

Let $g(x) = f^{-1}(x)$, the inverse of function f. Write the rule for g for each function f given below.

12. $f(x) = 3x - 8$

13. $f(x) = \frac{1}{2}x + 5$

14. $f(x) = 5x - 6$

15. $f(x) = -x + 7$

16. $f(x) = 7^x$

17. $f(x) = e^x$

18. $f(x) = \log_{12} x$

19. $f(x) = \ln x$

Simplify each expression.

20. $\log_9 9^x$

21. $15^{\log_{15} x}$

22. $\ln e^x$

23. $8^{\log_8 x}$

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Answers

9. Interchange x and y in the original function, and then solve for y. $x = 7y + 8$, $x - 8 = 7y$, $y = \frac{x-8}{7}$ or $f^{-1}(x) = \frac{x-8}{7}$

10. The line of symmetry for a function and its inverse is $x = y$.

11 a. False; it means the inverse of function f.

b. true

c. False; the original is not necessarily a function.

ACTIVITY 23 Continued

7–8 Create Representations, Note Taking, Debriefing Item 7 generalizes the results of Item 6. Tell students that these statements are considered *identities* for logarithms because the result is the same as the input. Have students add these to their math journals.

In Item 8, students apply the identities from Item 7. In part d, some students may need to be reminded that for common logarithms, it is not necessary to write the base of 10 as a subscript; it is assumed to be base 10.

Developing Math Language

By now, students are familiar with many mathematical concepts related to inverses, including inverse functions and inverse operations. The Inverse Properties of Logarithms are similar—they represent an "undoing" process.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the relationship between a function and its inverse. Students should be able to describe the relationship between the graph of a function and its inverse; they should also understand the composition of a function and its inverse.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 23-1 PRACTICE

12. $g(x) = \frac{x+8}{3}$ **13.** $g(x) = 2x - 10$

14. $g(x) = \frac{x+6}{5}$ **15.** $g(x) = -x + 7$

16. $g(x) = \log_7 x$ **17.** $g(x) = \ln x$

18. $g(x) = 12^x$ **19.** $g(x) = e^x$

20. x **21.** x

22. x **23.** x

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand the inverse relationship between logarithmic and exponential functions. Students should be able to write the inverse of a linear, exponential, or logarithmic function. As needed, provide students with extra practice in the form of a table or puzzle requiring students to match functions and their inverses.

Lesson 23-2

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–3
Check Your Understanding
#6 #7–8
Check Your Understanding
#11 #12 #13
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to simplify each expression.

1. $m^{10}(m^5)$ [m^{15}]
2. $(m^{10})^5$ [m^{50}]
3. $\frac{m^{10}}{m^5}$ [m^5]
4. $(m^{10})^{\frac{1}{2}}$ [m^5]

Universal Access

Some students may need review work with exponents before working on Items 1–3. While they are likely to remember how to work with positive integer exponents, they may need to be reminded that when $b \neq 0$, $b^0 = 1$ and $b^{-n} = \frac{1}{b^n}$.

Ask students to evaluate each expression.

a. 4^3 [64] b. 4^{-3} [$\frac{1}{64}$]
c. $\left(\frac{1}{4}\right)^{-3}$ [64] d. 4^0 [1]

1–3 Create Representations, Guess and Check, Debriefing The focus of Items 1 and 2 is for students to write equivalent forms of exponential and logarithmic statements.

For Item 3, suggest that they set each expression equal to x and then rewrite the equations in exponential form. They can then see that they are looking for a missing exponent. For example: $\log_2 32 = x \rightarrow 2^x = 32$, and so $x = 5$. They can guess and check to find the correct exponent.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand that a logarithm is an exponent. Discuss why a logarithm may not have a negative base.

6 Debriefing Students validate the Product, Quotient, and Power Properties of logarithms numerically.

My Notes

Learning Targets:

- Apply the properties of logarithms in any base.
- Compare and expand logarithmic expressions.
- Use the Change of Base Formula.

SUGGESTED LEARNING STRATEGIES: Create Representations, Close Reading

When rewriting expressions in exponential and logarithmic form, it is helpful to remember that a *logarithm is an exponent*. The exponential statement $2^3 = 8$ is equivalent to the logarithmic statement $\log_2 8 = 3$.

Notice that the logarithmic expression is equal to 3, which is the exponent in the exponential expression.

1. Express each exponential statement as a logarithmic statement.
   a. $3^4 = 81$ $\log_3 81 = 4$
   b. $6^{-2} = \frac{1}{36}$ $\log_6\left(\frac{1}{36}\right) = -2$
   c. $e^0 = 1$ $\ln 1 = 0$

2. Express each logarithmic statement as an exponential statement.
   a. $\log_4 16 = 2$ $4^2 = 16$
   b. $\log_5 125 = 3$ $5^3 = 125$
   c. $\ln 1 = 0$ $e^0 = 1$

3. Evaluate each expression without using a calculator.
   a. $\log_2 32$ 5
   b. $\log_4\left(\frac{1}{64}\right)$ -3
   c. $\log_3 27$ 3
   d. $\log_{12} 1$ 0

MATH TIP

Remember that a logarithm is an exponent. To evaluate the expression $\log_6 36$, find the exponent for 6 that gives the value 36. $6^2 = 36$. Therefore, $\log_6 36 = 2$.

Check Your Understanding

4. Why is the value of $\log_{-2} 16$ undefined?
5. **Critique the reasoning of others.** Mike said that the $\log_3$ of $\frac{1}{9}$ is undefined, because $3^{-2} = \frac{1}{9}$, and a log cannot have a negative value. Is Mike right? Why or why not?

The *Product, Quotient,* and *Power Properties* of common logarithms also extend to bases other than base 10.

6. Use the given property to rewrite each expression as a single logarithm. Then evaluate each logarithm in the equation to see that both sides of the equation are equal.
   a. **Product Property:** $\log_2 4 + \log_2 8 = \underline{\log_2 32}$
      $\underline{2} + \underline{3} = \underline{5}$
   b. **Quotient Property:** $\log_3 27 - \log_3 3 = \underline{\log_3 9}$
      $\underline{3} - \underline{1} = \underline{2}$
   c. **Power Property:** $2 \log_5 25 = \underline{\log_5 625}$
      $2 \cdot \underline{2} = \underline{4}$

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Answers

4. The value of b must be greater than 0, so $\log_{-2} 16$ is undefined.
5. Mike is mistaken; a logarithm can have a negative value, but it cannot have a negative base.

Lesson 23-2
Properties of Logarithms and the Change of Base Formula

ACTIVITY 23
continued

My Notes

7. Expand each expression. Assume all variables are positive.

a. $\log_7\left(\frac{x}{y^3}\right)$ **$\log_7 x - 3\log_7 y$**

b. $\log_4 x^2y$ **$2\log_4 x + \log_4 y$**

c. $\ln\left(\frac{x^2}{y^3}\right)$ **$2\ln x - 3\ln y$**

8. Assume that x is any real number, and decide whether the statement is *always true, sometimes true,* or *never true*. If the statement is sometimes true, give the conditions for which it is true.

a. $\log 7 - \log 5 = \frac{\log 7}{\log 5}$ **never true**

b. $\log_5 5^x = x$ **always true**

c. $2^{\log_2 x^2} = x^2$ **sometimes true, when $x \neq 0$**

d. $\log_4 3 + \log_4 5 - \log_4 x = \log_4 15$ **sometimes true, when $x = 1$**

e. $2\ln x = \ln x + \ln x$ **sometimes true, when $x > 0$**

Check Your Understanding

9. Attend to precision. Why is it important to specify the value of the variables as positive when using the Product, Quotient, and Power Properties of logarithms? Use Item 7 to state an example.

10. Simplify the following expression:
$\log 7 - \log 5$

Sometimes it is useful to change the base of a logarithmic expression. For example, the log key on a calculator is for common, or base 10, logs. Changing the base of a logarithm to 10 makes it easier to work with logarithms on a calculator.

11. Use the common logarithm function on a calculator to find the numerical value of each expression. Write the value in the first column of the table. Then write the numerical value using logarithms in base 2 in the second column.

	Numerical Value	$\log_2 a$
$\frac{\log 2}{\log 2}$	1	$\log_2 2$
$\frac{\log 4}{\log 2}$	**2**	**$\log_2 4$**
$\frac{\log 8}{\log 2}$	**3**	**$\log_2 8$**
$\frac{\log 16}{\log 2}$	**4**	**$\log_2 16$**
$\frac{\log N}{\log 2}$		**$\log_2 N$**

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ACTIVITY 23 Continued

7–8 Quickwrite, Debriefing In Item 7, students apply the Product, Quotient, and Power Properties to expand logarithmic expressions.

In Item 8a, point out that $\log 7 - \log 5 = \log\frac{7}{5} \neq \frac{\log 7}{\log 5}$. Students may mistakenly believe that this statement is always true, rather than never true. Item 8b is an identity that holds true for all x. Item 8c is sometimes true. It is true for all real numbers except 0. When $x = 0$, the left side of the equation is undefined and the right side is equal to 0. Item 8d is sometimes true. It is only true when $x = 1$, and therefore $\log_4 x = 0$. Part e is sometimes true. It is only true when x is greater than 0.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the Product, Quotient, and Power Properties for logarithms with bases other than 10. Ask students to write these properties using natural logarithms.

Answers

9. Suppose $y < 0$, in $\log_4 x^2y$. Then x^2y would be negative, and the logarithm of a negative number is undefined because $b > 0$.

10. $\log\frac{7}{5}$

11 Create Representations, Note Taking, Look for a Pattern Students can use a calculator to evaluate common logarithmic expressions of the form $\frac{\log\ (N)}{\log\ (2)}$ and then look for a pattern to find an equivalent expression of the form $\log_2 N$.

12 Create Representations, Note Taking, Look for a Pattern Students generalize the pattern from base 2 to base b. Students should add to their math journals the Change of Base Formula: $\log_b x = \frac{\log x}{\log b}$, where $b, x > 0$, $b \neq 1$.

13 Debriefing Students use the Change of Base Formula to approximate $\log_2 12$ on a calculator, first mentally approximating the value and then checking with a calculator to see that their answer is reasonable.

Differentiating Instruction

Extend students' knowledge of the Change of Base Formula. Item 12 shows the Change of Base Formula written in terms of the common logarithm because students can evaluate common logarithms on a calculator. However, this property of logarithms can be generalized to any base b: $\log_c a = \frac{\log_b a}{\log_b c}$, where a, b, $c > 0$ and $b, c \neq 0$. Challenge students to use the Change of Base Formula with a base other than 10 to evaluate the following logarithms without a calculator.

a. $\log_{16} 8$ [$\frac{3}{4}$; *change to base 2*]

b. $\log_9 27$ [$\frac{3}{2}$; *change to base 3*]

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the Change of Base Formula. Ask students to rewrite the Change of Base Formula using natural logarithms.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to rewrite exponential and logarithmic expressions in an equivalent form. As needed, have students practice creating equivalent expressions by writing each of the elements of an expression on a different index card or piece of paper. Then have students reposition the cards to form equivalent expressions.

ACTIVITY 23
continued

My Notes

Lesson 23-2
Properties of Logarithms and the Change of Base Formula

12. The patterns observed in the table in Item 11 illustrate the ***Change of Base Formula***. Make a conjecture about the Change of Base Formula of logarithms.

$\log_b x = \frac{\log x}{\log b}$

13. Consider the expression $\log_2 12$.

a. The value of $\log_2 12$ lies between which two integers? **3 and 4**

b. Write an equivalent common logarithm expression for $\log_2 12$, using the Change of Base Formula.
$\frac{\log(12)}{\log(2)}$

c. Use a calculator to find the value of $\log_2 12$ to three decimal places. Compare the value to your answer from part a.
3.585; The value lies between 3 and 4.

Check Your Understanding

14. Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.
a. $\log_5 32$ **b.** $\log_3 104$

15. In Item 13, how do you find out which values the value of $\log_2 12$ lies between?

LESSON 23-2 PRACTICE

Write a logarithmic statement for each exponential statement.

16. $7^3 = 343$

17. $3^{-2} = \frac{1}{9}$

18. $e^m = u$

Write an exponential statement for each logarithmic statement.

19. $\log_6 1296 = 4$

20. $\log_{\frac{1}{2}} 4 = -2$

21. $\ln x = t$

Evaluate each expression without using a calculator.

22. $\log_4 64$

23. $\log_2 \left(\frac{1}{32}\right)$

Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

24. $\log_3 7$

25. $\log_2 18$

26. $\log_{25} 4$

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Answers

14. a. $\frac{\log 32}{\log 5} \approx 2.153$

b. $\frac{\log 104}{\log 3} \approx 4.228$

15. Sample answer: 12 lies between two powers of 2, 8 and 16. $\log_2 8 = 3$; $\log_2 16 = 4$. Therefore, $\log_2 12$ lies between 3 and 4.

LESSON 23-2 PRACTICE

16. $\log_7 343 = 3$

17. $\log_3\left(\frac{1}{9}\right) = -2$

18. $\ln u = m$

19. $6^4 = 1296$

20. $\left(\frac{1}{2}\right)^{-2} = 4$

21. $e^t = x$

22. 3

23. -5

24. $\frac{\log 7}{\log 3} \approx 1.771$

25. $\frac{\log 18}{\log 2} \approx 4.170$

26. $\frac{\log 4}{\log 25} \approx 0.43$

My Notes

Learning Targets:

- Find intercepts and asymptotes of logarithmic functions.
- Determine the domain and range of a logarithmic function.
- Write and graph transformations of logarithmic functions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Close Reading, Quickwrite

1. Examine the function $f(x) = 2^x$ and its inverse, $g(x) = \log_2 x$.

a. Complete the table of data for $f(x) = 2^x$. Then use that data to complete a table of values for $g(x) = \log_2 x$.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$g(x) = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

b. Graph both $f(x) = 2^x$ and $g(x) = \log_2 x$ on the same grid.

c. What are the x- and y-intercepts for $f(x) = 2^x$ and $g(x) = \log_2 (x)$?
For $f(x) = 2^x$, the y-intercept is 1, and there is no x-intercept.
For $g(x) = \log_2 (x)$, the x-intercept is 1, and there is no y-intercept.

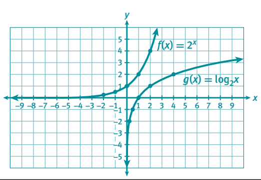
d. What is the line of symmetry between the graphs of $f(x) = 2^x$ and $g(x) = \log_2 x$? **$y = x$**

e. State the domain and range of each function using interval notation.

	Domain	Range
$f(x) = 2^x$	$(-\infty, \infty)$	$(0, \infty)$
$g(x) = \log_2 x$	$(0, \infty)$	$(-\infty, \infty)$

f. What is the end behavior of the graph of $f(x) = 2^x$?
As x approaches $-\infty$, y approaches 0. As x approaches ∞, y approaches ∞.

g. What is the end behavior of the graph of $g(x) = \log_2 (x)$?
As x approaches 0, y approaches $-\infty$. As x approaches ∞, y approaches ∞.



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ACTIVITY 23 Continued

Lesson 23-3

PLAN

Pacing: 1 class period

Chunking the Lesson

#1 #2
Check Your Understanding
#5 #6 #7
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Ask students to find any x- and y-intercepts of each of the following functions without graphing.

1. $f(x) = x^2 - 4$ [*x-intercepts: 2 and −2; y-intercept: −4*]
2. $f(x) = 3x - 12$ [*x-intercepts: 2 and −2; y-intercept: −4*]
3. $f(x) = (x - 1)^2 + 2$ [*no x-intercepts; y-intercept: 2*]
4. $f(x) = 10^x$ [*no x-intercept; y-intercept: 1*]

Teacher to Teacher

Lesson 23-3 investigates the graph of $g(x) = \log_2 x$ by relating it to the graph of the inverse function, $f(x) = 2^x$. Then transformations are used to graph functions of the form $g(x) = a \log_b (x - c) + d$ in Item 5.

1 Create Representations, Look for a Pattern, Debriefing After completing a table of values for $f(x) = 2^x$, students should realize they can interchange the x and y columns of the table to get some values for the inverse, $g(x) = \log_2 x$. Remind students of this relationship if necessary. The Math Tip on page 367 will reinforce this concept. Students then observe the symmetry of the graphs of f and g over the line $y = x$. They recognize that the domains and ranges of the two functions interchange, and that while one function has a horizontal asymptote at $y = 0$, the other has a vertical asymptote at $x = 0$.

2 Look for a Pattern, Think-Pair-Share, Debriefing Students perform an investigation similar to the work they did in Item 1, with base e instead of base 2. Have students discuss the similarities and differences in the answers to Items 1 and 2.

ELL Support

To support students' language acquisition, monitor their listening skills and understanding as they compare Items 1 and 2. Carefully pair students to ensure that all students participate in and learn from the discussion.

Technology Tip

If students do not have a natural logarithm key on their calculators, remind them that $\ln x = \frac{\log x}{\log e}$. Students can use the common logarithm key and a decimal approximation of e as an alternate method of evaluating natural logarithms.

For additional technology resources, visit SpringBoard Digital.

My Notes

TECHNOLOGY TIP

The [LN] key on your calculator is the natural logarithm key.

h. Write the equation of any asymptotes of each function.

$f(x) = 2^x$ ____ **$y = 0$** ____

$g(x) = \log_2 x$ ____ **$x = 0$** ____

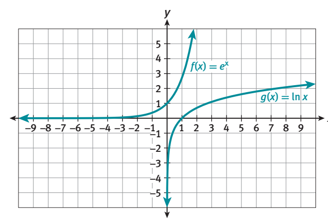
2. Examine the function $f(x) = e^x$ and its inverse, $g(x) = \ln x$.

a. Complete the table of data for $f(x) = e^x$. Then use those data to complete a table of values for $g(x) = \ln x$.

x	$f(x) = e^x$
−2	**0.135**
−1	**0.368**
0	**1**
1	**2.718**
2	**7.390**

x	$g(x) = \ln x$
0.135	**−2**
0.368	**−1**
1	**0**
2.718	**1**
7.390	**2**

b. Graph both $f(x) = e^x$ and $g(x) = \ln x$ on the same grid.



c. What are the x- and y-intercepts for $f(x) = e^x$ and $g(x) = \ln x$?
For $f(x) = e^x$, the y-intercept is 1, and there is no x-intercept.
For $g(x) = \ln x$, the x-intercept is 1, and there is no y-intercept.

d. What is the line of symmetry between the graphs of $f(x) = e^x$ and $g(x) = \ln x$? **$y = x$**

e. State the domain and range of each function using interval notation.

	Domain	**Range**
$f(x) = e^x$	**$(-\infty, \infty)$**	**$(0, \infty)$**
$g(x) = \ln x$	**$(0, \infty)$**	**$(-\infty, \infty)$**

f. What is the end behavior of the graph of $f(x) = e^x$?
As x approaches $-\infty$, y approaches 0.
As x approaches ∞, y approaches ∞.

g. What is the end behavior of the graph of $g(x) = \ln x$?
As x approaches 0, y approaches $-\infty$.
As x approaches ∞, y approaches ∞.

h. Write the equation of any asymptotes of each function.

$f(x) = e^x$ ____ **$y = 0$** ____

$g(x) = \ln x$ ____ **$x = 0$** ____

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Lesson 23-3
Graphs of Logarithmic Functions

My Notes

Check Your Understanding

3. Make sense of problems. From the graphs you drew for Items 1 and 2, draw conclusions about the behavior of inverse functions with respect to:
a. the intercepts
b. the end behavior
c. the asymptotes

4. If a function has an intercept of (0, 0), what point, if any, will be an intercept for the inverse function?

Transformations of the graph of the function $f(x) = \log_b x$ can be used to graph functions of the form $g(x) = a \log_b (x - c) + d$, where $b > 0$, $b \neq 1$. You can draw a quick sketch of each parent graph, $f(x) = \log_b x$, by plotting the points $\left(\frac{1}{b}, -1\right)$, (1, 0), and (b, 1).

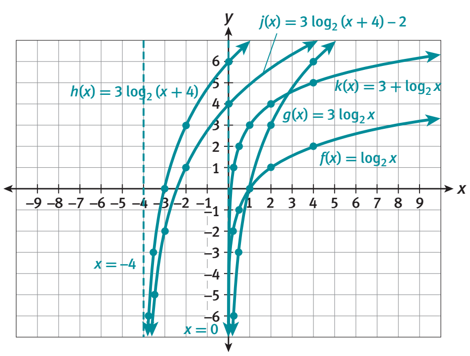
5. Sketch the parent graph $f(x) = \log_2 x$ on the axes below. Then, for each transformation of f, provide a verbal description and sketch the graph, including asymptotes.

a. $g(x) = 3 \log_2 x$
vertically stretch the graph of f by a factor of 3; asymptote $x = 0$

b. $h(x) = 3 \log_2 (x + 4)$
horizontally translate the graph of g left 4 units; asymptote $x = -4$

c. $j(x) = 3 \log_2 (x + 4) - 2$
vertically translate the graph of h down 2 units; asymptote $x = -4$

d. $k(x) = \log_2 (8x)$ **$k(x) = \log_2 8 + \log_2 x = 3 + \log_2 x$, so the graph of $\log_2 x$ shifts 3 units up; asymptote $x = 0$**



6. Explain how the function $j(x) = 3 \log_2 (x + 4) - 2$ can be entered on a graphing calculator using the common logarithm key. Then graph the function on a calculator and compare the graph to your answer in Item 5c. **Use the Change of Base Formula and rewrite $j(x) = 3 \log_2 (x + 4) - 2$ as $j(x) = \frac{3 \log (x + 4)}{\log (2)} - 2$.**

MATH TIP

Recall that a graph of the exponential function $f(x) = b^x$ can be drawn by plotting the points $\left(-1, \frac{1}{b}\right)$, (0, 1), and (1, b).

Switching the x- and y-coordinates of these points gives you three points on the graph of the inverse of $f(x) = b^x$, which is $f(x) = \log_b x$.

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ACTIVITY 23 Continued

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the behavior of the graphs of inverse exponential and logarithmic functions. Ask students to explain the relationship between the domains and ranges of inverse functions and how this relationship supports the answers to Item 3.

Answers

3. a. If the function has a y-intercept of 1, then the inverse function has an x-intercept of 1.
b. If the function approaches 0 on the y-axis as x approaches infinity, then in the inverse function, y approaches infinity as x approaches 0.
c. If the function has an asymptote at $x = 0$, then the inverse has an asymptote at $y = 0$.

4. (0, 0)

5 Activating Prior Knowledge, Create Representations, Quickwrite, Debriefing The Math Tip shows students why they can get a quick sketch of the graph of $g(x) = \log_2 x$ by plotting the points $\left(\frac{1}{2}, -1\right)$, (1, 0), and (2, 1). Students are expected to use their prior knowledge of graphing other functions by transformations to sketch transformations of the logarithmic function.

6 Quickwrite Students can use the Change of Base Formula in order to graph the function on a calculator.

7 Group Presentation, Debriefing, Paraphrasing Monitor group presentations on the effects of parameters to ensure that students are communicating clearly and that they are using terms such as *stretch*, *shrink*, *reflection*, and *translation* correctly. After group presentations and a debriefing, have students summarize the information about transformations in their math journals.

Check Your Understanding

Debrief students' answers to this item to ensure that they connect transformations of logarithmic functions with their prior knowledge of transformations of quadratic functions. If students have difficulty generalizing the effects of *a* and *c*, illustrate with specific examples of quadratic and logarithmic functions.

Answers

8. a. Both cause a vertical stretch or shrink by a factor of *a*.
b. Both cause a horizontal translation *c* units to the right.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 23-3 PRACTICE

9. The *x*-intercept is 1, and there is no *y*-intercept.
10. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand graphing logarithmic and exponential functions. As needed, provide students with additional practice using transformations to graph functions. Students may benefit from additional practice graphing quadratic functions or square root functions to better understand the effect of the parameters on the graph of the parent function.

My Notes

Lesson 23-3
Graphs of Logarithmic Functions

7. Explain how the parameters *a*, *c*, and *d* transform the parent graph $f(x) = \log_b x$ to produce a graph of the function $g(x) = a \log_b (x - c) + d$.
Transformations of the parent graph $f(x) = \log_b x$ that produce the graph of $g(x) = a \log_b (x - c) + d$ are described below.

$|a| > 1 \Rightarrow$ **a vertical stretch by a factor of** a
$0 < |a| < 1 \Rightarrow$ **a vertical shrink by a factor of** a
$a < 0 \Rightarrow$ **a reflection over the x-axis**
$c > 0 \Rightarrow$ **a horizontal translation right c units**
$c < 0 \Rightarrow$ **a horizontal translation left c units**
$d < 0 \Rightarrow$ **a vertical shift down d units**
$d > 0 \Rightarrow$ **a vertical shift up d units**

Check Your Understanding

8. Look for and make use of structure.
a. Compare the effect of *a* in a logarithmic function $a \log_b x$ to *a* in a quadratic function ax^2 (assume *a* is positive).
b. Compare the effect of *c* in a logarithmic function $\log_b (x - c)$ to *c* in a quadratic function $(x - c)^2$.

LESSON 23-3 PRACTICE

9. Given an exponential function that has a *y*-intercept of 1 and no *x*-intercept, what is true about the intercepts of the function's inverse?

10. Make sense of problems. The inverse of a function has a domain of $(-\infty, \infty)$ and a range of $(0, \infty)$. What is true about the original function's domain and range?

Model with mathematics. Graph each function, using a parent graph and the appropriate transformations. Describe the transformations.

11. $f(x) = 2 \log_2 (x) - 6$
12. $f(x) = \log_3 (x - 5) + 1$
13. $f(x) = \frac{1}{2} \log_4 x$
14. $f(x) = \log_2 (x + 4) - 3$

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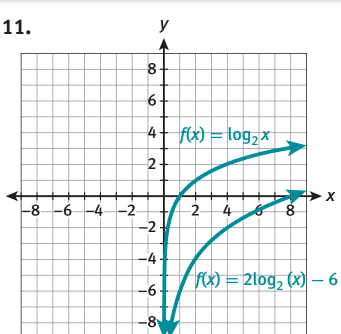
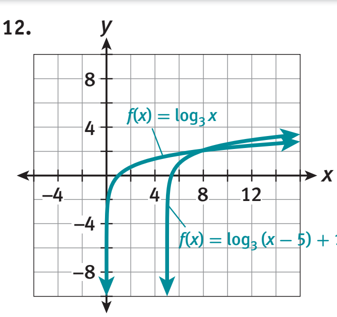
11.



vertically stretch the parent graph by a factor of 2, and then vertically translate down 6 units

12.



horizontally translate the parent graph right 5 units, and then vertically translate up 1 unit

Inverse Functions: Exponential and Logarithmic Functions
Undoing It All

ACTIVITY 23
continued

ACTIVITY 23 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 23-1

Let $g(x) = f^{-1}(x)$, the inverse of function f. Write the rule for g for each function f given below.

1. $f(x) = 7x - 9$
2. $f(x) = \left(\frac{1}{3}\right)^x$
3. $f(x) = 2x - 8$
4. $f(x) = -x + 3$
5. $f(x) = 5^x$
6. $f(x) = e^x$
7. $f(x) = \log_{20} x$
8. $f(x) = \ln x$

Simplify each expression.

9. $\log_3 3^x$
10. $12^{\log_{12} x}$
11. $\ln e^x$
12. $7^{\log_7 x}$

Lesson 23-2

Express each exponential statement as a logarithmic statement.

13. $12^2 = 144$
14. $2^{-3} = \frac{1}{8}$
15. $e^n = m$
16. $e^{3x} = 2$
17. $10^2 = 100$
18. $e^0 = 1$

Express each logarithmic statement as an exponential statement.

19. $\log_3 9 = 2$
20. $\log_2 64 = 6$
21. $\ln 1 = 0$
22. $\ln x = 6$
23. $\log_2 64 = 6$
24. $\ln e = 1$

Expand each expression. Assume all variables are positive.

25. $\log_2 x^2y^5$
26. $\log_4\left(\frac{x^8}{5}\right)$
27. $\ln ex$
28. $\ln\left(\frac{1}{x}\right)$
29. Which is an equivalent form of the expression $\ln 5 + 2\ln x$?
A. $5\ln x^2$
B. $\ln 2x^5$
C. $\ln 5x^2$
D. $2\ln x^5$

ACTIVITY PRACTICE

1. $g(x) = \frac{x+9}{7}$
2. $g(x) = \log_{\frac{1}{3}}(x)$
3. $g(x) = \frac{x+8}{2}$
4. $g(x) = 3 - x$
5. $g(x) = \log_5 x$
6. $g(x) = \ln x$
7. $g(x) = 20^x$
8. $g(x) = e^x$
9. x
10. x
11. x
12. x
13. $\log_{12} 144 = 2$
14. $\log_2\left(\frac{1}{8}\right) = -3$
15. $\ln m = n$
16. $\ln 2 = 3x$
17. $\log 100 = 2$
18. $\ln 1 = 0$
19. $3^2 = 9$
20. $2^6 = 64$
21. $e^0 = 1$
22. $e^6 = x$
23. $2^6 = 64$
24. $e^1 = e$
25. $2\log_2 x + 5\log_2 y$
26. $8\log_4 x + \log_4 5$
27. $\ln e + \ln x$
28. $\ln 1 - \ln x$
29. C
30. $\log_2 64$
31. $\log_3 \frac{x^2}{y}$
32. $\ln 2x$
33. $\ln x^3$
34. 1
35. 3
36. 2
37. 4
38. $\frac{\log 20}{\log 4} \approx 2.161$
39. $\frac{\log 4}{\log 20} \approx 0.463$
40. $\frac{\log 45}{\log 5} \approx 2.365$
41. $\frac{\log 18}{\log 3} \approx 2.631$
42. D

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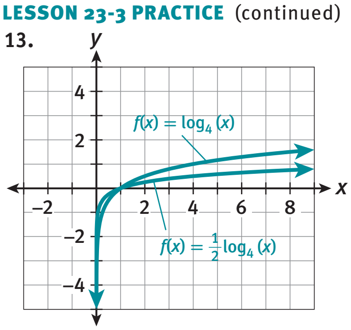
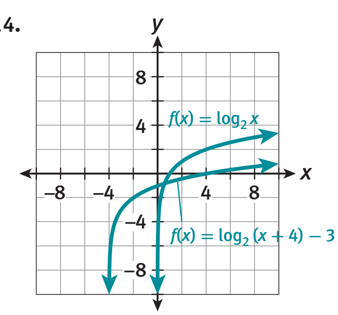
LESSON 23-3 PRACTICE (continued)

13.



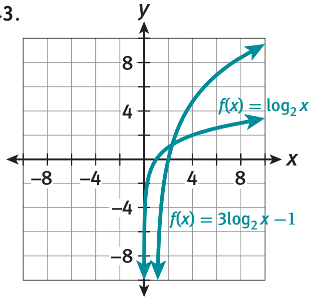
vertically shrink the parent graph by a factor of $\frac{1}{2}$

14.



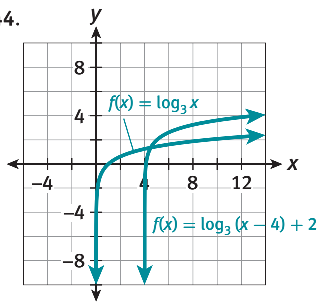
horizontally translate the parent graph left 4 units, and then vertically translate down 3 units

43.



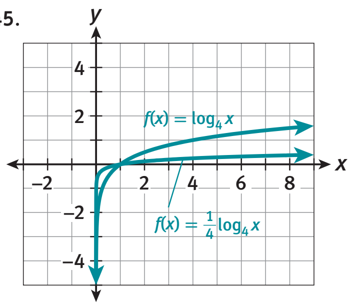
vertically stretch parent graph by a factor of 3, and then translate vertically down 1 unit

44.



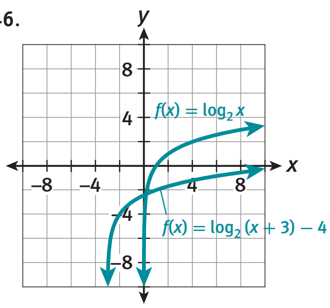
horizontally translate the parent graph right 4 units, and then vertically translate up 2 units

45.



vertically shrink the parent graph by a factor of $\frac{1}{4}$

46.



horizontally translate the parent graph 3 left, and then vertically translate down 4 units

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

Rewrite each expression as a single, simplified logarithmic term. Assume all variables are positive.

30. $\log_2 32 + \log_2 2$

31. $\log_3 x^2 - \log_3 y$

32. $\ln x + \ln 2$

33. $3 \ln x$

Evaluate each expression without using a calculator.

34. $\log_{12} 12$

35. $\log_7 343$

36. $\log_7 49$

37. $\log_3 81$

Change each expression to a logarithmic expression in base 10. Use a calculator to find the value to three decimal places.

38. $\log_4 20$

39. $\log_{20} 4$

40. $\log_5 45$

41. $\log_3 18$

Lesson 23-3

42. If the domain of a logarithmic function is $(0, \infty)$ and the range is $(-\infty, \infty)$, what are the domain and range of the inverse of the function?

A. domain: $(-\infty, \infty)$, range: $(-\infty, \infty)$
B. domain: $(0, \infty)$, range: $(-\infty, \infty)$
C. domain: $(-\infty, \infty)$, range: $(-\infty, \infty)$
D. domain: $(-\infty, \infty)$, range: $(0, \infty)$

Graph each function, using a parent graph and the appropriate transformations. Describe the transformations.

43. $f(x) = 3 \log_2 (x) - 1$

44. $f(x) = \log_3 (x - 4) + 2$

45. $f(x) = \frac{1}{4} \log_4 x$

46. $f(x) = \log_2 (x + 3) - 4$

MATHEMATICAL PRACTICES
Model with Mathematics

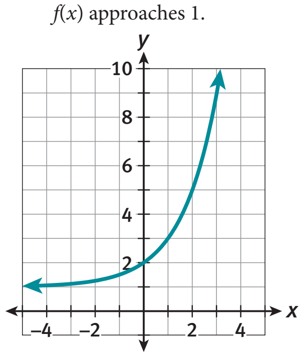
47. Given the function $f(x) = 2^x + 1$

a. Give the domain, range, y-intercept, and any asymptotes for $f(x)$. Explain.

b. Draw a sketch of the graph of the function on a grid. Describe the behavior of the function as x approaches ∞ and as x approaches $-\infty$.

47. a. domain: $(-\infty, \infty)$; range: $(1, \infty)$; y-intercept: 2; asymptote: $y = 1$

b. As x approaches ∞, $f(x)$ approaches ∞; as x approaches $-\infty$, $f(x)$ approaches 1.



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Logarithmic and Exponential Equations and Inequalities

ACTIVITY 24

College Costs

Lesson 24-1 Exponential Equations

My Notes

Learning Targets:

- Write exponential equations to represent situations.
- Solve exponential equations.

SUGGESTED LEARNING STRATEGIES: Summarizing, Paraphrasing, Create Representations, Vocabulary Organizer, Note Taking, Group Presentation

Wesley is researching college costs. He is considering two schools: a four-year private college where tuition and fees for the current year cost about $24,000, and a four-year public university where tuition and fees for the current year cost about $10,000. Wesley learned that over the last decade, tuition and fees have increased an average of 5.6% per year in four-year private colleges and an average of 7.1% per year in four-year public colleges.

To answer Items 1–4, assume that tuition and fees continue to increase at the same average rate per year as in the last decade.

1. Complete the table of values to show the estimated tuition for the next four years.

Years from Present	Private College Tuition and Fees	Public College Tuition and Fees
0	$24,000	$10,000
1	**$25,344**	**$10,710**
2	**$26,763.26**	**$11,470.41**
3	**$28,262.01**	**$12,284.81**
4	**$29,844.68**	**$13,157.03**

2. **Express regularity in repeated reasoning.** Write two functions to model the data in the table above. Let $R(t)$ represent the private tuition and fees and $U(t)$ represent the public tuition and fees, where t is the number of years from the present.
   $R(t) = 24{,}000(1.056)^t$
   $U(t) = 10{,}000(1.071)^t$

3. Wesley plans to be a senior in college six years from now. Use the models above to find the estimated tuition and fees at both the private and public colleges for his senior year in college.
   $R(6) = \$33{,}280.88$
   $U(6) = \$15{,}091.65$

4. **Use appropriate tools strategically.** Write an equation that can be solved to predict the number of years that it will take for the public college tuition and fees to reach the current private tuition and fees of $24,000. Find the solution using both the graphing and table features of a calculator. **$10{,}000(1.071)^t = 24{,}000$; graphing each side of the equation gives the intersection point (12.763, 24,000) and a table shows the values below. Therefore, it will take 13 years.**

t	U
12	22,776
13	24,393

MATH TIP

To solve an equation graphically on a calculator, enter each side of the equation as a separate function and find the intersection point of the two functions.

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Common Core State Standards for Activity 24

HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*

HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

HSA-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*

ACTIVITY 24

Directed

Activity Standards Focus

In this activity, students explore exponential and logarithmic equations and solve them using properties of exponents and logarithms. They will also use technology to approximate the solutions of exponential and logarithmic equations using tables of values and graphing. Students will also investigate and learn how to solve exponential and logarithmic inequalities.

Lesson 24-1

PLAN

Pacing: 1 class period

Chunking the Lesson

#1–3 #4

Example A Example B

Check Your Understanding

Lesson Practice

TEACH

Bell-Ringer Activity

Provide students with a list of numbers that can each be expressed in exponential form with base 2. Have students rewrite these numbers in exponential form.

1–3 Create Representations, Debriefing Tell students to keep all decimal places in their calculations, but to record answers to two decimal places. Some students may first find the amount of increase each year and then add it to the previous year's amount. Other students may recognize that this situation represents exponential growth and use the growth factor to complete the table. Have students share the different methods they use to complete the table. Students should realize that the multiplicative pattern indicates an exponential function for the model.

Some students may evaluate the functions at $t = 6$ from Item 2 to answer Item 3. Other students may extend the pattern in the table.

4 Create Representations, Debriefing Students will write an exponential equation and solve it graphically and numerically. To solve an equation graphically on the calculator, students can enter the left side of their equation as one function and the right side as a second function, choose an appropriate window, and then find the point of intersection of the two functions. To solve an equation numerically, students can enter the function U into a graphing calculator and then scroll down a table to find when the value of U first exceeds $24,000.

Example A, Example B Note Taking, Group Presentation Debriefing One method of solving exponential equations is shown in these examples. The equation can be rewritten so that there are like bases on each side of the equation. To use this method, students will need to understand that $b^m = b^n$ if and only if $m = n$.

Have the students work together in groups to solve the Try These items. Encourage students to check their answers by substituting the solution into the original equation. Invite students to share their strategies for rewriting numbers in exponential form.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand how to solve exponential equations using the property that $b^m = b^n$ if and only if $m = n$.

Answers

5. Look at the bases on each side of the equation and see if you can write one as a power of the other.
6. Substitute your solution into the original equation; $3^4 - 1 = 80$; $81 - 1 = 80$; $80 = 80$.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

7. $\frac{1}{2}$
8. 2
9. 3
10. -2
11. 2
12. 7
13. 2
14. 6
15. No; 2 and 27 cannot be written using the same base.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve exponential equations when the equation can be rewritten so that there is the same base on both sides of the equation. Guide students to practice rewriting numbers in exponential form, and reinforce the rule: $b^m = b^n$ if and only if $m = n$. Remind students that these types of exponential equations are a special case and that this method will not solve all exponential equations.

ACTIVITY 24 *continued*

My Notes

Lesson 24-1
Exponential Equations

Solving a problem like the one in Item 4 involves solving an exponential equation. An ***exponential equation*** is an equation in which the variable is in the exponent. Sometimes you can solve an exponential equation by writing both sides of the equation in terms of the same base. Then use the fact that when the bases are the same, the exponents must be equal:

$$b^m = b^n \text{ if and only if } m = n$$

Example A

Solve $6 \cdot 4^x = 96$.

$6 \cdot 4^x = 96$

Step 1: $4^x = 16$ — Divide both sides by 6.

Step 2: $4^x = 4^2$ — Write both sides in terms of base 4.

Step 3: $x = 2$ — If $b^m = b^n$, then $m = n$.

Example B

Solve $5^{4x} = 125^{x-1}$.

$5^{4x} = 125^{x-1}$

Step 1: $5^{4x} = (5^3)^{x-1}$ — Write both sides in terms of base 5.

Step 2: $5^{4x} = 5^{3x-3}$ — Power of a Power Property: $(a^m)^n = a^{mn}$

Step 3: $4x = 3x - 3$ — If $b^m = b^n$, then $m = n$.

Step 4: $x = -3$ — Solve for x.

MATH TIP

Check your work by substituting your solutions into the original problem and verifying the equation is true.

Try These A–B

Solve for x. Show your work.

a. $3^x - 1 = 80$ 4

b. $2^x = \frac{1}{32}$ -5

c. $6^{3x-4} = 36^{x+1}$ 6

d. $\left(\frac{1}{7}\right)^x = \left(\frac{1}{49}\right)$ 2

Check Your Understanding

5. When writing both sides of an equation in terms of the same base, how do you determine the base to use?
6. How could you check your solution to an exponential equation? Show how to check your answers to Try These part a.

LESSON 24-1 PRACTICE

Make use of structure. Solve for x by writing both sides of the equation in terms of the same base.

7. $2^{10x} = 32$
8. $4^x - 5 = 11$
9. $2^{4x-2} = 4^{x+2}$
10. $8^x = \frac{1}{64}$
11. $4 \cdot 5^x = 100$
12. $3 \cdot 2^x = 384$
13. $\left(\frac{1}{3}\right)^{2x} = \left(\frac{1}{9}\right)^{4-x}$
14. $\left(\frac{1}{2}\right)^{2x} = \left(\frac{1}{8}\right)^{10-x}$
15. Can you apply the method used in this lesson to solve the equation $2^{4x} = 27$? Explain why or why not.

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Common Core State Standards for Activity 24 *(continued)*

HSA-REI.D.11 Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

HSF-LE.A.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Lesson 24-2
Solving Equations by Using Logarithms

Learning Targets:
- Solve exponential equations using logarithms.
- Estimate the solution to an exponential equation.
- Apply the compounded interest formula.

SUGGESTED LEARNING STRATEGIES: Note Taking, Group Presentation, Create Representations, Close Reading, Vocabulary Organizer

For many exponential equations, it is not possible to rewrite the equation in terms of the same base. In this case, use the concept of inverses to solve the equation symbolically.

Example A
Estimate the solution of $3^x = 32$. Then solve to three decimal places.
Estimate that x is between 3 and 4, because $3^3 = 27$ and $3^4 = 81$.

$3^x = 32$

Step 1: $\log_3 3^x = \log_3 32$ — Take the log base 3 of both sides.

Step 2: $x = \log_3 32$ — Use the Inverse Property to simplify the left side.

Step 3: $x = \dfrac{\log 32}{\log 3}$ — Use the Change of Base Formula.

Step 4: $x \approx 3.155$ — Use a calculator to simplify.

Try These A
Estimate each solution. Then solve to three decimal places. Show your work.

a. $6^x = 12$ between 1 and 2; $x = \dfrac{\log 12}{\log 6} \approx 1.387$

b. $5^x = 610$ between 3 and 4; $x = \dfrac{\log 610}{\log 5} \approx 3.985$

c. $4^x = 0.28$ between -1 and 0; $x = \dfrac{\log 0.28}{\log 4} \approx -0.918$

d. $e^x = 91$ between 4 and 5; $x = \ln 91 \approx 4.512$

Example B
Find the solution of $4^{x-2} = 35.6$ to three decimal places.

$4^{x-2} = 35.6$

$\log_4 4^{x-2} = \log_4 35.6$ — Take the log base 4 of both sides.

Step 1: $x - 2 = \log_4 35.6$ — Use the Inverse Property to simplify the left side.

Step 2: $x = \log_4 35.6 + 2$ — Solve for x.

Step 3: $x = \dfrac{\log 35.6}{\log 4} + 2$ — Use the Change of Base Formula.

Step 4: $x \approx 4.577$ — Use a calculator to simplify.

Try These B
Find each solution to three decimal places. Show your work.

a. $12^{x+3} = 240$ $x = \dfrac{\log 240}{\log 12} - 3 \approx -0.794$

b. $4.2^{x+4} + 0.8 = 5.7$ $x = \dfrac{\log 4.9}{\log 4.2} - 4 \approx -2.893$

c. $e^{2x-4} = 148$ $x = \dfrac{\ln 148 + 4}{2} \approx 4.499$

My Notes

MATH TIP

Recall that the *Inverse Properties* of logarithms state that for $b > 0, b \neq 1$:

$$\log_b b^x = x$$

and

$$b^{\log_b x} = x$$

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ACTIVITY 24 Continued
Lesson 24-2

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A
Example B
#1
Example C
Example D
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Continue the Bell-Ringer Activity from Lesson 24-1 by providing students with a new list of numbers. This time, include numbers that have different bases when written in exponential form. For example, your list might include 32, 81, 100, 150, 225, 243, and 400.

Example A, Example B Note Taking, Group Presentation These examples deal with exponential equations that cannot be written in terms of the same base. In this case, students will use logarithms to solve the exponential equations. Students are often taught to simply rewrite an exponential equation in logarithmic form in order to solve the equation. When doing this, the concept of an exponential function base b and the logarithmic function base b as inverses is often lost. This is an important concept in later mathematics, such as solving some separable differential equations in calculus. Therefore, it is a good idea to include Step 1 of the solution in each example.

As you model Example A, have students approximate logarithms by deciding which two integers the value lies between. Then, when students evaluate the logarithms on their calculators, they will know if their answers are reasonable.

As students work through the Try These items, circulate from group to group, asking and answering questions in order to assist students as needed. Then have them present their work to the entire class.

1 Create Representations, Debriefing Students return to Item 4 in Lesson 24-1 to solve the exponential equation analytically. Their work is validated when they see the solution is the same as their graphical and numerical solutions.

Developing Math Language

As students respond to questions or discuss possible solutions to problems, monitor their use of the vocabulary *compound interest* and *continuously compounded interest* to ensure their understanding and ability to use language correctly and precisely.

Example C Create Representations, Debriefing Compound interest problems provide students with an example of an application of exponential growth.

Differentiating Instruction

Some students may need examples in order to apply the compound interest formula correctly. Use the items below for that purpose.

a. If you deposit $500 in an account paying 3.5% annual interest compounded semiannually, how much money will be in the account after 5 years?

$$\left[A = 500\left(1+\frac{0.035}{2}\right)^{2.5} \approx \$594.72\right]$$

b. How long will it take for an investment of $2500 to earn $500 interest in an account that pays 4% annual interest compounded quarterly?

$$[A = 2500 + 500 = 3000$$

$$3000 = 2500\left(1+\frac{0.04}{4}\right)^{(4t)}$$

$$1.2 = 1.01^{4t}$$

$$\log_{101} 1.2 = \log_{101} 101^{4t}$$

$$\log_{101} 1.2 = 4t$$

$$t = \left(\frac{1}{4}\right)\left(\frac{\log 1.2}{\log 101}\right) \approx 4.581$$

4 years and 9 months]

My Notes

MATH TERMS

Compound interest is interest that is earned or paid not only on the principal but also on previously accumulated interest. At specific periods of time, such as daily or annually, the interest earned is added to the principal and then earns additional interest during the next period.

MATH TIP

When interest is compounded annually, it is paid once a year. Other common compounding times are shown below.

Times per Year	
Semiannually	2
Quarterly	4
Monthly	12
Weekly	52
Daily	365

1. Rewrite the equation you wrote in Item 4 of Lesson 24-1. Then show how to solve the equation using the Inverse Property.

$10{,}000\,(1.071)^t = 24{,}000$

$1.071^t = 2.4$

$\log_{1.071} 1.071^t = \log_{1.071} 2.4$

$t = \dfrac{\log 2.4}{\log 1.071}$

$t \approx 12.763$

Wesley's grandfather gave him a birthday gift of $3000 to use for college. Wesley plans to deposit the money in a savings account. Most banks pay ***compound interest***, so he can use the formula below to find the amount of money in his savings account after a given period of time.

Compound Interest Formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = amount in account
P = principal invested
r = annual interest rate as a decimal
n = number of times per year that interest is compounded
t = number of years

Example C

If Wesley deposits the gift from his grandfather into an account that pays 4% annual interest compounded quarterly, how much money will Wesley have in the account after three years?

Substitute into the compound interest formula. Use a calculator to simplify.

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 3000\left(1+\frac{0.04}{4}\right)^{4(3)} \approx \$3380.48$$

Solution: Wesley will have $3380.48 in the account after three years.

Try These C

How long would it take an investment of $5000 to earn $1000 interest if it is invested in a savings account that pays 3.75% annual interest compounded monthly?

4.869 years ⇒ 4 years 11 months

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MINI-LESSON: Compounding Periods and Finding Time

If students need additional help calculating compound interest, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

Lesson 24-2
Solving Equations by Using Logarithms

ACTIVITY 24
continued

My Notes

Wesley's grandfather recommends that Wesley deposit his gift into an account that earns interest compounded continuously, instead of at a fixed number of times per year.

Continuously Compounded Interest Formula

$A = Pe^{rt}$

- A = amount in account
- P = principal invested
- r = annual interest rate as a decimal
- t = number of years

Example D
If Wesley deposits the gift from his grandfather into an account that pays 4% annual interest compounded continuously, how much money will Wesley have in the account after three years?

Substitute into the continuously compounded interest formula. Use a calculator to simplify.

$$A = Pe^{rt} = 3000e^{0.04(3)} \approx \$3382.49$$

Solution: Wesley will have $3382.49 in the account after three years.

Try These D
How long would it take an investment of $5000 to earn $1000 interest if it is invested in a savings account that pays 3.75% annual interest compounded continuously?

4.862 years

Check Your Understanding

2. How is solving exponential and logarithmic equations similar to other equations that you have solved?

3. **Attend to precision.** In Examples C and D, why are the answers rounded to two decimal places?

4. A bank advertises an account that pays a monthly interest rate of 0.3% compounded continuously. What value do you use for r in the continuously compounded interest formula? Explain.

LESSON 24-2 PRACTICE
Solve for x to three decimal places.

5. $8^x = 100$

6. $3^{x-4} = 85$

7. $3e^{x+2} = 87$

8. $2^{3x-2} + 7 = 25$

9. $2 \cdot 4^{3x} - 3 = 27$

10. $e^{2x} - 1.5 = 6.7$

11. **Make sense of problems.** A deposit of $4000 is made into a savings account that pays 2.48% annual interest compounded quarterly.
- **a.** How much money will be in the account after three years?
- **b.** How long will it take for the account to earn $500 interest?
- **c.** How much more money will be in the account after three years if the interest is compounded continuously?

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LESSON 24-2 PRACTICE

5. 2.215
6. 8.044
7. 1.367
8. 2.057
9. 0.651
10. 1.052
11. a. $4307.96
b. 4.764 years
c. $0.99

ACTIVITY 24 Continued

Example D Note Taking, Group Presentation, Debriefing Invite students to discuss what it means to compound interest continuously. Students can calculate compound interest over shorter and shorter time periods to investigate continuously compounded interest. The idea that functions have limiting values is foundational in calculus.

Exponential growth, like the growth of an investment through compound interest, is a key application of exponential functions. After completing the Try These items, invite students to suggest other examples of exponential growth.

Check Your Understanding
Debrief students' answers to these items to ensure that they understand how to solve exponential equations. Students should also be able to apply the appropriate formulas for compound interest.

Answers

2. Similar to other equations, solving exponential and logarithmic equations requires performing inverse operations on both sides of the equation.

3. The answers represent amounts of money, which are always rounded to the nearest cent.

4. 3.6% or 0.036; The monthly interest rate must be multiplied by 12 to get an annual rate to use in the formula.

ASSESS
Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT
Check students' answers to the Lesson Practice to ensure that they understand how to solve exponential equations using logarithms. Monitor students' work to ensure that they are applying the Change of Base Formula for logarithms correctly. Encourage students to estimate each answer before using technology.

Lesson 24-3

PLAN

Pacing: 1 class period

Chunking the Lesson

Example A
Example B
Example C
Example D
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity

Provide students with several expressions that can be simplified using the laws of logarithms. For example: $2 \log x + 3 \log y$. Students should rewrite each expression as a single logarithm.

Developing Math Language

Students should add the definitions of *logarithmic equation* and *extraneous solution* to their math journals. Students can use their prior knowledge to understand the meanings of these terms.

Example A Group Presentation, Note Taking, Debriefing The next set of examples cover equations with one logarithmic expression and equations with two logarithmic expressions in the same base set equal to each other. Focus on Step 1 in Example A, again stressing the concept that the exponential function base b and the logarithmic function base b are inverses. Some students may see that you also can get to Step 2 by writing the original equation in exponential form.

After reviewing the Try These items, emphasize the importance of careful, correct notation. Encourage students to show their work with every step, reminding them that writing the steps is often more efficient than trying to keep track of too many steps mentally.

My Notes

MATH TERMS

An **extraneous solution** is a solution that arises from a simplified form of the equation that does not make the original equation true.

Learning Targets:

- Solve logarithmic equations.
- Identify extraneous solutions to logarithmic equations.
- Use properties of logarithms to rewrite logarithmic expressions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Vocabulary Organizer, Note Taking, Group Presentation

Equations that involve logarithms of variable expressions are called ***logarithmic equations***. You can solve some logarithmic equations symbolically by using the concept of functions and their inverses. Since the domain of logarithmic functions is restricted to the positive real numbers, it is necessary to check for ***extraneous solutions*** when solving logarithmic equations.

Example A

Solve $\log_4 (3x - 1) = 2$.

$\log_4 (3x - 1) = 2$

Step 1: $4^{\log_4 (3x-1)} = 4^2$ — Write in exponential form using 4 as the base.

Step 2: $3x - 1 = 16$ — Use the Inverse Property to simplify the left side.

Step 3: $x = \frac{17}{3}$ — Solve for x.

Check: $\log_4 (3 \cdot \frac{17}{3} - 1) = \log_4 16 = 2$

Try These A

Solve for x. Show your work.

a. $\log_3 (x - 1) = 5$
$3^{\log_3 (x-1)} = 3^5$
$x - 1 = 243$
$x = 244$

b. $\log_2 (2x - 3) = 3$
$2^{\log_2 (2x-1)} = 2^3$
$2x - 3 = 8$
$x = 5.5$

c. $4 \ln (3x) = 8$
$e^{\ln(3x)} = e^{2s}$
$3x = e^2$
$x = \frac{e^2}{3}$

To solve other logarithmic equations, use the fact that when the bases are the same, $m > 0$, $n > 0$, and $b \neq 1$, the logarithmic values must be equal:

$$\log_b m = \log_b n \text{ if and only if } m = n$$

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Lesson 24-3
Logarithmic Equations

ACTIVITY 24 *continued*

My Notes

Example B

Solve $\log_3(2x-3) = \log_3(x+4)$.

$\log_3(2x-3) = \log_3(x+4)$

Step 1: $2x - 3 = x + 4$ — If $\log_b m = \log_b n$, then $m = n$.

Step 2: $x = 7$ — Solve for x.

Check: $\log_3(2 \cdot 7 - 3) \stackrel{?}{=} \log_3(7+4)$

$\log_3 11 = \log_3 11$

Try These B

Solve for x. Check for extraneous solutions. Show your work.

a. $\log_6(3x+4) = 1$

b. $\log_5(7x-2) = \log_5(3x+6)$

c. $\ln 10 - \ln(4x-6) = 0$

a. $3x + 4 = 6^1$; $3x = 2$; $x = \frac{2}{3}$

b. $7x - 2 = 3x + 6$; $4x = 8$; $x = 2$

c. $10 = 4x - 6$; $16 = 4x$; $x = 4$

Sometimes it is necessary to use properties of logarithms to simplify one side of a logarithmic equation before solving the equation.

Example C

Solve $\log_2 x + \log_2(x+2) = 3$.

$\log_2 x + \log_2(x+2) = 3$

Step 1: $\log_2[x(x+2)] = 3$ — Product Property of Logarithms

Step 2: $2^{\log_2[x(x+2)]} = 2^3$ — Write in exponential form using 2 as the base.

Step 3: $x(x+2) = 8$ — Use the Inverse Property to simplify.

Step 4: $x^2 + 2x - 8 = 0$ — Write as a quadratic equation.

Step 5: $(x+4)(x-2) = 0$ — Solve the quadratic equation.

Step 6: $x = -4$ or $x = 2$ — Check for extraneous solutions.

Check: $\log_2(-4) + \log(-4+2) \stackrel{?}{=} 3$

$\log_2(-4) + \log(-2) \stackrel{?}{=} 3$

$\log_2 2 + \log(2+2) \stackrel{?}{=} 3$

$\log_2 2 + \log 4 \stackrel{?}{=} 3$

$\log_2 8 \stackrel{?}{=} 3$

$3 = 3$

Because $\log_2(-4)$ and $\log(-2)$ are not defined, -4 is not a solution of the original equation; thus it is extraneous.

The solution is $x = 2$.

Try These C

Solve for x, rounding to three decimal places if necessary. Check for extraneous solutions.

a. $\log_4(x+6) - \log_4 x = 2$ $\quad \frac{2}{5}$

b. $\ln(2x+2) + \ln 5 = 2$ $\quad -0.261$

c. $\log_2 2x + \log_2(x-3) = 3$ $\quad$ 4; -1 is extraneous

ACTIVITY 24 Continued

Example B Note Taking, Group Presentation In order to understand Example B, students will need to grasp the fact that $\log_b m = \log_b n$ if and only if $m = n$.

Have students work in groups on the Try These B items. Remind students that they must check solutions to logarithmic equations to identify any that may be extraneous.

Example C Note Taking Focus students on using logarithmic properties to simplify terms in the logarithmic equation before solving. This example also provides a problem with an extraneous solution.

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Example D Note Taking, Think-Pair-Share Explain that, in Example D, you are introducing the variable y. Use the transitive property of equality to state $-x = y$ and $y = \log x$. Guide students to understand that the solution of this set of equations is the solution of the original equation.

Technology Tip

The Try These D equations provide students with an opportunity to explore the use of graphs and tables to approximate the solution of logarithmic equations. For additional technology resources, visit SpringBoard Digital.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the concept of extraneous solutions for logarithmic equations. If students struggle with the concept of extraneous solutions for logarithmic equations, demonstrate the parallel with extraneous solutions of radical equations.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve logarithmic equations. Students who need additional practice approximating solutions can apply graphing techniques to solve linear or quadratic equations. Provide students with examples that they can solve algebraically. Guide them to approximate the solutions graphically and then to confirm the approximation by finding the solution algebraically.

My Notes

Some logarithmic equations cannot be solved symbolically using the previous methods. A graphing calculator can be used to solve these equations.

Example D

Solve $-x = \log x$ using a graphing calculator.

$-x = \log x$

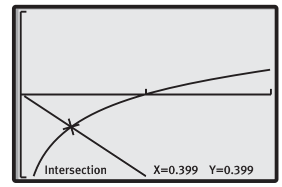
Step 1: Enter $-x$ for Y1.

Step 2: Enter $\log x$ for Y2.

Step 3: Graph both functions.

Step 4: Find the x-coordinate of the point of intersection: $x \approx 0.399$

Solution: $x \approx 0.399$



Try These D

Solve for x.

a. $x \log x = 3$
$x \approx 4.556$

b. $\ln x = -x^2 - 1$
$x \approx 0.330$

c. $\ln(2x + 4) = x^2$
$x \approx -0.89$ or 1.38

Check Your Understanding

1. Explain how it is possible to have more than one solution to a simplified logarithmic equation, only one of which is valid.

2. Critique the reasoning of others. Than solves a logarithmic equation and gets two possible solutions, -2 and 4. Than immediately decides that -2 is an extraneous solution, because it is negative. Do you agree with his decision? Explain your reasoning.

LESSON 24-3 PRACTICE

Solve for x, rounding to three decimal places if necessary. Check for extraneous solutions.

3. $\log_5(3x + 4) = 2$

4. $\log_3(4x + 1) = 4$

5. $\log_{12}(4x - 2) = \log_{12}(x + 10)$

6. $\log_2 3 + \log_2(x - 4) = 4$

7. $\ln(x + 4) - \ln(x - 4) = 4$

8. Construct viable arguments. You saw in this lesson that logarithmic equations may have extraneous solutions. Do exponential equations ever have extraneous solutions? Justify your answer.

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Answers

1. Sample answer: Solving a logarithmic equation can require simplifying expressions to a quadratic equation having two solutions. Because the input of a logarithm must be a positive real number, one solution may not satisfy the original equation.

2. No; Than needs to check both solutions. A solution is extraneous only if it causes a logarithm to have an input that is not positive.

LESSON 24-3 PRACTICE

3. 7

4. 20

5. 4

6. $\frac{28}{3} \approx 9.333$

7. 4.149

8. No; logarithmic equations sometimes have extraneous solutions because the domain of a logarithmic function is limited to positive numbers, so you may not be able to substitute a solution back into the original equation if it results in taking the logarithm of a negative number. However, exponential functions have a domain of all real numbers, so no solutions will be extraneous.

Lesson 24-4
Exponential and Logarithmic Inequalities

Learning Targets:
- Solve exponential inequalities.
- Solve logarithmic inequalities.

SUGGESTED LEARNING STRATEGIES: Note Taking, Group Presentation, Create Representations

You can use a graphing calculator to solve exponential and logarithmic inequalities.

My Notes

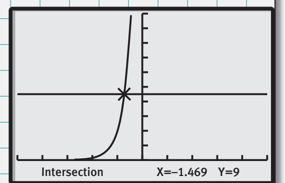
Example A
Use a graphing calculator to solve the inequality $4.2^{x+3} > 9$.

Step 1: Enter 4.2^{x+3} for Y1 and 9 for Y2.

Step 2: Find the x-coordinate of the point of intersection: $x \approx -1.469$

Step 3: The graph of $y = 4.2^{x+3}$ is above the graph of $y = 9$ when $x > -1.469$.

Solution: $x > -1.469$



Try These A
Use a graphing calculator to solve each inequality.

a. $3 \cdot 5.1^{1-x} < 75$ $x > -0.978$

b. $\log 10x \geq 1.5$ $x \geq 3.162$

c. $7.2 \ln x + 3.9 \leq 12$ $0 < x \leq 3.080$

Example B
Scientists have found a relationship between atmospheric pressure and altitudes up to 50 miles above sea level that can be modeled by $P = 14.7(0.5)^{\frac{a}{3.6}}$. P is the atmospheric pressure in lb/in.^{2} Solve the equation $P = 14.7(0.5)^{\frac{a}{3.6}}$ for a. Use this equation to find the atmospheric pressure when the altitude is greater than 2 miles.

Step 1: Solve the equation for a.

$$\frac{P}{14.7} = 0.5^{\frac{a}{3.6}}$$ Divide both sides by 14.7.

$$\log_{0.5}\left(\frac{P}{14.7}\right) = \log_{0.5}\left(0.5^{\frac{a}{3.6}}\right)$$ Take the log base 0.5 of each side.

$$\log_{0.5}\left(\frac{P}{14.7}\right) = \frac{a}{3.6}$$ Simplify.

$$3.6\log_{0.5}\left(\frac{P}{14.7}\right) = a$$ Multiply both sides by 3.6.

$$\frac{3.6\log\left(\frac{P}{14.7}\right)}{\log 0.5} = a$$ Use the Change of Base Formula.

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Universal Access
Students who find it challenging to solve exponential or logarithmic functions algebraically will benefit from the technology tools discussed in this lesson. These tools allow students to concentrate on guiding principles without getting confused by exponential or logarithmic notation. Guide these students to connect the algebraic solutions to those found using graphs and tables.

ACTIVITY 24 Continued
Lesson 24-4

PLAN
Pacing: 1 class period

Chunking the Lesson
Example A
Example B
Check Your Understanding
Lesson Practice

TEACH
Bell-Ringer Activity
Have students solve the following system of inequalities.

$$x^2 + 3x + 1 > y$$
$$4x + 5y > 20$$

Invite students to share their strategies for solution.

Teacher to Teacher
The items in this lesson give students the opportunity to work with exponential and logarithmic inequalities.

Example A Create Representations, Think-Pair-Share, Group Presentation
Demonstrate how to use the table feature of the graphing calculator to solve the exponential inequality. Adjust the value of Δy to find more precise approximations.

Have students work in their groups to complete the Try These A items. Within each group, have some students use tables of values and other students approximate solutions graphically. Students then compare their methods and their solutions. Invite students to share their methods and strategies with the class.

Technology Tip
In order to use the graphing feature of their calculators to approximate equation solutions, students must define each function that the calculator will graph. These functions are then available for use with the table feature. For additional technology resources, visit SpringBoard Digital.

Example B Create Representations
Students solve the inequality $14.7(0.5)^{\frac{x}{3.6}} < 5$ by setting each side of the equation equal to y and then graphing these equations, finding the intersection, and selecting the interval where the exponential function is below the linear function. There is also a restriction that the model holds for $x < 50$, so the solution is $5.601 < x < 50$.

Example B (continued) In Try These B Item a, students should solve the inequality $20 < -400 + 180 \log x < 30$. They can graph $y = 20$, $y = 30$, and $y = -400 + 180 \log x$ and find where the logarithmic function occurs between the two linear functions. Students will need to find an appropriate viewing window. One possible window is $0 < x < 300$ with a scale of 25 and $0 < y < 40$ with a scale of 10.

Check Your Understanding

Debrief students' answers to these items to ensure that they understand the methods used to solve exponential and logarithmic inequalities. Invite students to share their responses and discuss their favorite strategies for solving inequalities.

Answers

2. Sample answer: The answer is a range of values instead of a single value.
3. Sample answer: Graph both sides of the inequality. When one graph is above the other graph, then that side of the inequality is greater than the other side. This will occur on one side of the intersection point. Find the x-value of the intersection point and use the appropriate inequality symbol(s) to write the solution.

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

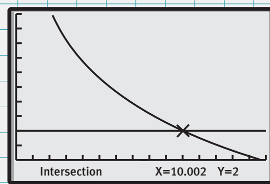
See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 24-4 PRACTICE

4. $x \le 5.052$
5. $2.562 < x < 10.095$
6. $x \le -0.812$
7. $x > 11.057$

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand how to solve exponential and logarithmic inequalities. Students who need additional practice may benefit from applying the techniques of this lesson to review solving linear or quadratic inequalities.



Step 2: Use your graphing calculator to solve the inequality
$$\frac{3.6 \log\left(\frac{P}{14.7}\right)}{\log 0.5} > 2.$$

The graph of $y = \dfrac{3.6 \log\left(\frac{x}{14.7}\right)}{\log 0.5}$ is above the graph of $y = 2$ when $0 < x < 10.002$.

Solution: When the altitude is greater than 2, the atmospheric pressure is between 0 and 10.002 lb/in.^{2}.

Try These B

Suppose that the relationship between C, the number of digital cameras supplied, and the price x per camera in dollars is modeled by the function $C = -400 + 180 \cdot \log x$.

a. Find the range in the price predicted by the model if there are between 20 and 30 cameras supplied.
from \$215.44 to \$244.84

b. Solve the equation for x. Use this equation to find the number of cameras supplied when the price per camera is more than \$300.
$x = 10^{\frac{C+400}{180}}$**; more than 45 cameras**

Check Your Understanding

2. How are exponential and logarithmic inequalities different from exponential and logarithmic equations?
3. Describe how to find the solution of an exponential or logarithmic inequality from a graph. What is the importance of the intersection point in this process?

LESSON 24-4 PRACTICE

Use a graphing calculator to solve each inequality.

4. $16.4(0.87)^{x-1.5} \ge 10$
5. $30 < 25 \log(3.5x - 4) + 12.6 < 50$
6. $4.5e^x \le 2$
7. $\ln(x - 7.2) > 1.35$

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Logarithmic and Exponential Equations and Inequalities
College Costs

ACTIVITY 24 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 24-1

1. Which exponential equation can be solved by rewriting both sides in terms of the same base?
   A. $4^x = 12$
   B. $6 \cdot 2^{x-3} = 256$
   C. $3^{x+2} - 5 = 22$
   D. $e^x = 58$
2. Solve for x.
   a. $16^x = 32^{x-1}$
   b. $8 \cdot 3^x = 216$
   c. $5^x = \frac{1}{625}$
   d. $7^{2x} = 343^{x-4}$
   e. $4^x + 8 = 72$
   f. $e^x = 3$
   g. $e^{3x} = 2$
   h. $3e^{5x} = 42$

Lesson 24-2

3. Solve for x to three decimal places.
   a. $7^x = 300$
   b. $5^{x-4} = 135$
   c. $3^{2x+1} - 5 = 80$
   d. $3 \cdot 6^{3x} = 0.01$
   e. $5^x = 212$
   f. $3(2^{x+4}) = 350$
4. A deposit of $1000 is made into a savings account that pays 4% annual interest compounded monthly.
   a. How much money will be in the account after 6 years?
   b. How long will it take for the $1000 to double?
5. June invests $7500 at 12% interest for one year.
   a. How much would she have if the interest is compounded yearly?
   b. How much would she have if the interest is compounded daily?
6. If $4000 is invested at 7% interest per year compounded continuously, how long will it take to double the original investment?
7. At what annual interest rate, compounded continuously, will money triple in nine years?
   A. 1.3%
   B. 7.3%
   C. 8.1%
   D. 12.2%

Lesson 24-3

8. Compare the methods of solving equations in the form of *log* = *log* (such as $\log_3(2x - 3) = \log_3(x + 4)$) and *log* = *number* (such as $\log_4(3x - 1) = 2$).
9. Solve for x. Check for extraneous solutions.
   a. $\log_2(5x - 2) = 3$
   b. $\log_4(2x - 3) = 2$
   c. $\log_7(5x + 3) = \log_7(3x + 11)$
   d. $\log_6 4 + \log_6(x + 2) = 1$
   e. $\log_3(x + 8) = 2 - \log_3(x)$
   f. $\log_2(x + 6) - \log_2(x) = 3$
   g. $\log_2 x - \log_2 5 = \log_2 10$
   h. $5 \ln 3x = 40$
   i. $\ln 4x = 30$

ACTIVITY PRACTICE

1. C
2. **a.** 5
   b. 3
   c. −4
   d. 12
   e. 3
   f. ln 3
   g. $\frac{1}{3}\ln 2$
   h. $\frac{1}{5}\ln 14$
3. **a.** 2.931
   b. 7.048
   c. 1.522
   d. −1.061
   e. 3.328
   f. 2.866
4. **a.** $1270.74
   b. 17.358 years
5. **a.** $8400.00
   b. $8456.06
6. 9.9 years
7. D
8. To find a solution of a *log* = *log* equation, use the fact that when the bases are the same and $b, m, n > 0$, $b \neq 1$, the logarithmic values must be equal: $\log_b m = \log_b n$ if and only if $m = n$. For *log* = *number*, you can write in exponential form and solve.
9. **a.** 2
   b. $\frac{19}{2}$
   c. 4
   d. $-\frac{1}{2}$
   e. 1
   f. $\frac{6}{7}$
   g. 50
   h. $\frac{1}{3}e^8$
   i. $\frac{1}{4}e^{30}$

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10. a. The solutions for $\log(x-2)$ must be greater than 2, because any solutions equal to or less than 2 will result in the log of a negative number.
b. The solutions for $\log(x+3)$ must be greater than -3, because any solutions equal to or less than -3 will result in the log of a negative number.

11. a. 1.939
b. 0.788; 5.256

12. a. $0.611 < x < 1.473$
b. $x \geq 0.753$
c. $x \geq 6$
d. $0 < x \leq 1$
e. $x \leq -2.24$

13. a. $y_1(1000) \approx 2.71692$; $y_1(10{,}000) \approx 2.71815$; $y_1(1{,}000{,}000) \approx 2.71828$
b. The value of the expression gets closer and closer to e.
c. Since $m = \frac{n}{r}$, $\frac{r}{n} = \frac{1}{m}$ and $n = mr$. Therefore,
$$A = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{1}{m}\right)^{mrt} = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}.$$
d. Sample answer: The compounded interest formula may be written as $A = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$, but as the number of compounding periods (n) increases, m also increases, and the expression in square brackets gets closer and closer to e. So it makes sense that the formula for continuously compounded interest is $A = Pe^{rt}$.

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

10. If an equation contains
a. $\log(x-2)$, how do you know the solutions must be greater than 2?
b. $\log(x+3)$, how do you know solutions must be greater than -3?

11. Solve for x to three decimal places using a graphing calculator.
a. $\ln 3x = x^2 - 2$
b. $\log(x+7) = x^2 - 6x + 5$

Lesson 24-4

12. Use a graphing calculator to solve each inequality.
a. $2000 < 1500(1.04)^{12x} < 3000$
b. $4.5\log(2x) + 8.4 \geq 9.2$
c. $\log_3(3x-5) \geq \log_3(x+7)$
d. $\log_2 2x \leq \log_4(x+3)$
e. $5^{x+3} \leq 2^{x+4}$

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

13. Explore how the compounded interest formula is related to the continuously compounded interest formula.
a. Consider the expression $\left(1+\frac{1}{m}\right)^{m}$, where m is a positive integer. Enter the expression in your calculator as y_1. Then find the value of $y_1(1000)$, $y_1(10{,}000)$, and $y_1(1{,}000{,}000)$.
b. As m increases, what happens to the value of the expression?
c. The compounded interest formula is $A = P\left(1+\frac{r}{n}\right)^{nt}$. Let $m = \frac{n}{r}$. Explain why the formula may be written as $A = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$.
d. As the number of compounding periods, n, increases, so does the value of m. Explain how your results from parts b and c show the connection between the compounded interest formula and the continuously compounded interest formula.

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Exponential and Logarithmic Equations

EVALUATING YOUR INTEREST

Embedded Assessment 3
Use after Activity 24

1. **Make use of structure.** Express each exponential statement as a logarithmic statement.
   a. $5^{-3} = \frac{1}{125}$ **b.** $7^2 = 49$
   c. $20^2 = 400$ **d.** $3^6 = 729$
2. Express each logarithmic statement as an exponential statement.
   a. $\log_8 512 = 3$ **b.** $\log_9\left(\frac{1}{729}\right) = -3$
   c. $\log_2 64 = 6$ **d.** $\log_{11} 14{,}641 = 4$
3. Evaluate each expression without using a calculator.
   a. $25^{\log_{25} x}$ **b.** $\log_3 3^x$ **c.** $\log_3 27$
   d. $\log_8 1$ **e.** $\log_2 40 - \log_2 5$ **f.** $\frac{\log 25}{\log 5}$
4. Solve each equation symbolically. Give approximate answers rounded to three decimal places. Check your solutions. Show your work.
   a. $4^{2x-1} = 64$ **b.** $5^x = 38$
   c. $3^{x+2} = 98.7$ **d.** $2^{3x-4} + 7.5 = 23.6$
   e. $\log_3(2x + 1) = 4$ **f.** $\log_8(3x - 2) = \log_8(x + 1)$
   g. $\log_2(3x - 2) + \log_2 8 = 5$ **h.** $\log_6(x - 5) + \log_6 x = 2$
5. Let $f(x) = \log_2(x - 1) + 3$.
   a. Sketch a parent graph and a series of transformations that result in the graph of f.
   b. Give the equation of the vertical asymptote of the graph of f.
6. **Make sense of problems.** Katie deposits $10,000 in a savings account that pays 8.5% interest per year, compounded quarterly. She does not deposit more money and does not withdraw any money.
   a. Write the formula to find the amount in the account after 3 years.
   b. Find the total amount she will have in the account after 3 years.
7. How long would it take an investment of $6500 to earn $1200 interest if it is invested in a savings account that pays 4% annual interest compounded quarterly? Show the solution both graphically and symbolically.

Common Core State Standards for Embedded Assessment 3

HSA-CED.A.1	Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*
HSA-CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
HSF-IF.C.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
HSF-IF.C.7e	Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
HSF-LE.A.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

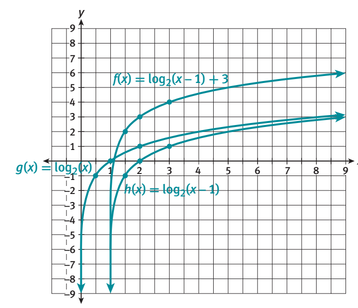
Embedded Assessment 3

Assessment Focus

- Solving exponential equations
- Solving logarithmic equations
- Solving real-world applications of exponential and logarithmic functions

Answer Key

1. **a.** $\log_5\left(\frac{1}{125}\right) = -3$ **b.** $\log_7 49 = 2$
   c. $\log_{20} 400 = 2$ **d.** $\log_3 729 = 6$
2. **a.** $8^3 = 512$ **b.** $9^{-3} = \frac{1}{729}$
   c. $2^6 = 64$ **d.** $11^4 = 14{,}641$
3. **a.** x **b.** x **c.** 3
   d. 0 **e.** 3 **f.** 2
4. **a.** 2 **b.** 2.260
   c. 2.180 **d.** 2.670
   e. 40 **f.** 1.5
   g. 2 **h.** 9
5. **a.**



   b. $x = 1$
6. **a.** $A = 10{,}000\left(1 + \frac{0.085}{4}\right)^{(4)(3)}$
   b. $12,870.19
7. 4.256 years; Check students' graphs. Solution should appear as point of intersection of $f(x) = 1.01^{4x}$ and $f(x) = 1.1846$.

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Teacher to Teacher

You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Embedded Assessment 3

Use after Activity 24

Exponential and Logarithmic Equations

EVALUATING YOUR INTEREST

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1–7)	• Fluency and accuracy in evaluating and rewriting exponential and logarithmic equations and expressions • Effective understanding of and accuracy in solving logarithmic and exponential equations algebraically and graphically • Effective understanding of logarithmic functions and their key features as transformations of a parent graph	• Largely correct work when evaluating and rewriting exponential and logarithmic equations and expressions • Adequate understanding of how to solve logarithmic and exponential equations algebraically and graphically leading to solutions that are usually correct • Adequate understanding of logarithmic functions and their key features as transformations of a parent graph	• Difficulty when evaluating and rewriting logarithmic and exponential equations and expressions • Partial understanding of how to solve logarithmic and exponential equations algebraically and graphically • Partial understanding of logarithmic functions and their key features as transformations of a parent graph	• Mostly inaccurate or incomplete work when evaluating and rewriting logarithmic and exponential equations and expressions • Inaccurate or incomplete understanding of how to solve exponential and logarithmic equations algebraically and graphically • Little or no understanding of logarithmic functions and their key features as transformations of a parent graph
Problem Solving (Items 6, 7)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 5–7)	• Fluency in modeling a real-world scenario with an exponential equation or graph • Effective understanding of how to graph a logarithmic function using transformations	• Little difficulty in accurately modeling a real-world scenario with an exponential equation or graph • Largely correct understanding of how to graph a logarithmic function using transformations	• Some difficulty in modeling a real-world scenario with an exponential equation or graph • Partial understanding of how to graph a logarithmic function using transformations	• Significant difficulty with modeling a real-world scenario with an exponential equation or graph • Mostly inaccurate or incomplete understanding of how to graph a logarithmic function using transformations
Reasoning and Communication (Items 6, 7)	• Clear and accurate use of mathematical work to justify an answer	• Correct use of mathematical work to justify an answer	• Partially correct justification of an answer using mathematical work	• Incorrect or incomplete justification of an answer using mathematical work

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Common Core State Standards for Embedded Assessment 3 *(cont.)*

HSF-BF.A.1	Write a function that describes a relationship between two quantities.*
HSF-BF.A.1c (+)	Compose functions.
HSF-BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
HSF-BF.B.4	Find inverse functions.
HSF-BF.B.4b (+)	Verify by composition that one function is the inverse of another.